

On Gibbs measure and weak flow for the cubic NLS with non-localised initial data

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Abstract

In this paper we prove the existence of an invariant measure for the cubic NLS

$$i\partial_t u + \Delta u - |u|^2 u = 0$$

on the real line in the sense that we prove the existence of a measure ρ supported by non-localised functions such that there exists random variables $X(t)$ whose laws are ρ (thus independent of t) and such that $t \mapsto X(t)$ is a solution to the cubic NLS. Our strategy for the proof is inspired by [8] and relies on the application of Prokhorov and Skorokhod Theorems to a sequence of measures which are invariant under some approximating flows, as we proved in our previous [12]. However, the work by Bourgain, [5] provides a stronger result than this one, as it gives almost sure strong solutions for the cubic NLS and the invariance of the measure can be deduced from it.

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1 Introduction

The problem of building invariant measures under nonlinear flows for PDEs dates back to the pioneering work of Lebowitz-Rose-Speer [20], and was subsequently addressed by Bourgain in his seminal papers on KdV and Schrödinger on the torus [4].

In the subsequent years, a lot of effort has been spent in order to investigate the connections between invariant measures, and more in general the effect of "randomization" in PDEs, with various properties of the corresponding flows. This kind of probabilistic approach has been indeed successfully developed in several contests to significantly improve the existing deterministic theory: among the most remarkable results, we mention the papers [10]-[11] in which these ideas are developed to prove some supercritical well-posedness for the cubic wave equation.

An attempt of providing a complete or even satisfying literature on the topic is out of our scope here, and we refer to the recent [26] for a fairly complete picture. We should anyway stress the fact that most of the available results concern evolution equations on compact manifolds. The reason for this is somehow technical, as having a countable basis of eigenfunctions for the Laplacian turns to be a major advantage (suggesting a "natural" randomization) in the construction of an invariant measure, which can be roughly summarized by the following scheme: frequency truncation-Liouville theorem-uniform probability estimates. Randomization (and related questions) in a non-compact setting turns in fact to be much more complicated, and is a topic currently attracting a lot of attention from the community, which has produced some significant results in the very last years. We mention [9] in which the authors consider a NLS on the real line with a well chosen external potential "trapping" the solution (see also [3] in which modulation spaces are used and [16] for the 2D case), [21] in which supercritical well posedness for NLW on \mathbb{R}^3 is discussed (see also [28] for 3D NLW).

Our recent paper [12] fits within this contest: there, we built a Gibbs measure for the cubic-type NLS

$$i\partial_t u - \Delta u + \chi|u|^2 u = 0, \quad u : \mathbb{R}_t \times \mathbb{R}_x \rightarrow \mathbb{C} \quad (1)$$

with a smooth interaction potential χ satisfying some smallness assumptions (namely, $0 \leq \chi \lesssim \langle x \rangle^{-\alpha}$ and $|(1 - \Delta)^{s_0/2} \chi| \lesssim \langle x \rangle^{-\alpha}$ for some $\alpha > 1$, $s_0 > 1/4$), and proved it to be invariant under the flow of the equation above on a suitable topological σ -algebra. The main novelty of [12] is represented by the randomization, as there is no trapping potential coming into play. Inspired by [22] (see also [15] for the Klein-Gordon equation), our randomization is therefore given by

$$\phi(x) \cong \int_{\mathbb{R}} \frac{e^{inx}}{\sqrt{1+n^2}} dW_n(\omega)$$

where ω is the random event and W_n is a Brownian motion, which makes $\phi(x)$ Itô integral. The presence of the function χ in (1) is essentially technical, and heavily used in the convergence argument (the strategy to build the invariant measure consists in approximating the flow of (1) with "approaching" equations on finite dimensional spaces, define invariant measures for them and then pass to the limit; this requires several tools from local and global deterministic analysis).

The purpose of this paper is essentially to show that the function χ in (1) can be removed, and thus to build a random variable which is a solution of the cubic NLS

$$i\partial_t u - \Delta u + |u|^2 u = 0, \quad u : \mathbb{R}_t \times \mathbb{R}_x \rightarrow \mathbb{C} \quad (2)$$

whose law does not depend on time.

Our strategy is inspired by [8], in which the authors develop a clever machinery relying on two important results of measure theory, Prokhorov and Skorokhod Theorems, to build an invariant measure for some different dispersive flows on compact manifolds and obtaining, as an application, existence of solutions for the corresponding equations in certain spaces. This strategy comes from fluid mechanics, see for instance [1, 13] and reference therein. In [8], the authors adapt it to dispersive equations. We follow their proof.

We briefly summarize the main steps needed (all the details and the definitions will be made clear through the paper):

1) Given a PDE and its associated flow $\Phi(t)$, introduce an approximating problem such that its corresponding flow $\Phi_N(t)$ is global and such that we are able to construct an invariant measure ρ_N on some topological space X , for which we have good control.

2) Define, for every N , the measure $\nu_N = \rho_N \circ \Phi_N^{-1}$ and show that it is *tight* in some space $C([-T, T]; Y)$ with $Y \supset X$. Therefore, the application of Prokhorov Theorem gives the existence of a measure ν on $C([-T, T]; Y)$ which is the weak limit of the sequence ν_N .

3) Apply Skorokhod Theorem to construct a sequence of random variables converging to a solution of the initial problem.

We will thus apply this strategy to our setting with the aim of removing the interaction potential χ in (1). Equation (1) will of course play the role of the "approximating problem", and we will use the invariant measure built in [12] (although slightly changing the topological setting) as the approximating ρ_N . Anyway, we remark that the main difference with [8] is in that we here work in a non compact setting: this will make the limiting argument significantly more complicated, and we will have to rely on some Feynman-Kac type results to make things work.

We stress that the main difference with [8] comes from the proof of the tightness of the family $(\nu_N)_N$. The tightness is deduced from uniform bounds on ρ_N and the invariance of ρ_N under the flow Φ_N . This does not change for us. Nevertheless, the uniform bounds on ρ_N in [8] is based on the fact that ρ_N can be written $d\rho_N(u) = F_N(u)dq(u)$ where q is a well-known measure, often a Brownian bridge, and F_N converges in $L^1(dq)$. This is not our case. The sequence of measure ρ_N converges for path integral reasons towards a measure ρ mutually singular with q , [25, 17]. One of the main problem is here to understand this convergence in order to get uniform bounds on the sequence $(\rho_N)_N$. This requires to use Feynman-Kac or integral paths results. So, the novelty of this work consists in putting together Feynman-Kac type results [25, 17] with the Prokhorov-Skorokhod machinery.

We are now ready to state our main result (we postpone to the next subsection the definition of the functional spaces).

Theorem 1. *There exist a probability space (Ω, \mathcal{F}, P) , a random variable $X = X(t)$ with values in $X_{\varphi, T}^\varepsilon$ and a measure ρ such that*

- *For every t , the law of $X(t)$ is ρ (thus, in particular, it does not depend on t);*
- *The random variable X is almost surely a weak solution to (2);*
- *The measure ρ is supported by non localised functions (not $L^2(\mathbb{R})$).*

Remark 1.1. *With this strategy of proof, we cannot state a stronger result such as there exists a flow $\psi(t)$ of (2) and a measure ρ such that ρ is invariant under $\psi(t)$ because the random variable $X(t)$ allows us only to define a weak flow of (2) which in particular is not necessarily unique. We discuss the link between uniqueness and invariance in the Subsection 4.5. Nevertheless, the measure ρ is formally written $e^{-\mathcal{E}(u)}du$ where $\mathcal{E}(u)$ is the energy plus the mass, which makes it*

close to a Gibbs measure. Hence, at least in the sense of the above theorem, we get invariance of the "Gibbs" measure.

However, in [5], Bourgain showed the uniqueness of the solution. Bourgain did not study the limiting measure, but its existence can be deduced from quantum field theory, such as in [25] as explained later or [17], Chapter 3 and more precisely Theorem 3.3.3. Besides, combining the existence of the measure and the uniform convergence theorem of Bourgain, one should be able to prove the strong invariance of the limiting measure under the flow hence defined, giving a much stronger result.

Remark 1.2. Considering only the kinetic part dq of the approximating measures

$$d\rho_N(u) = F_N(u)dq(u),$$

one can see that q is somehow a large box limit. Indeed, q is the limit of a sequence q_L , where q_L is the law of

$$\varphi_L(x) = \sum_{k \in \mathbb{Z}} L^{-1/2} e^{ikx/L} (1 + (k/L)^2)^{-1/2} g_k$$

with (g_k) independent centred normalised Gaussian variables. This limiting process has been explained in [12] and consists in building a Wiener integral. The random variable φ_L is built as a map of $2\pi L\mathbb{T}$ with $\varphi_L = (1 - \Delta)^{-1/2} \underline{\varphi}_L$ and

$$\underline{\varphi}_L(x) = \sum_{k \in \mathbb{Z}} \frac{e^{ikx/L}}{\sqrt{L}} g_k.$$

We note that $(x \mapsto \frac{e^{ikx/L}}{\sqrt{2\pi L}})_k$ is an orthonormal basis of $L^2(2\pi L\mathbb{T})$. The measure dq is obtained by letting the size of the box (or torus) $2\pi L$ go to ∞ . Besides, taking the mean value of the L^2 norm of φ_L to the square gives something of order L and hence diverges. This is a way to understand the non locality of the initial data.

However and as we have mentioned earlier, our final measure ρ is mutually singular with q . Nevertheless, thanks to Feynman-Kac type results, we know that ρ is invariant under translations and that when u has the law ρ , the law of $u(x)$ is (not depending on x and) absolutely continuous with regard to the Lebesgue measure. This is sufficient to prove that ρ is supported by non localised functions, as we see in Propositions 3.7, 3.8. In particular, ρ is not supported by functions which are not in $L^2(\mathbb{R})$.

Remark 1.3. Our proof is adaptable to other non-linearities. In fact, it is adaptable to other Hamiltonians. The sufficient conditions are given by the Feynman-Kac theory. But at least, one could consider a quintic non linearity. Or an equation with the same Hamiltonian but with a different dispersion relation. The fact that the measure is supported by non-localised functions may be explained by the fact that localised data may generate scattering. Then, the solutions would converge towards 0 for some norm, but, as we see in Subsection 3.4, invariance in a weaker norm often implies invariance in a stronger norm. This would contradict the invariance of the measure (it cannot both be invariant and converge to a Dirac delta centred in 0 when time goes to $\pm\infty$).

Let us give some details on the plan of the paper. In the next section, we will provide the necessary notations, introducing the functional spaces and the measures we will deal with. In section 3 we will review and discuss some known results that will be the main ingredients in

the proof of Theorem 1: in particular, we will recall some generalities on Feynman-Kac theory for oscillatory processes and Prokhorov and Skorokhod Theorems. In subsection 3.4 we will show how to adapt our previous result of [12] to the present functional setting. Section 4 will be devoted to the proof of our main Theorem, that will be divided in several steps. First of all, we shall state two technical results (Lemmas 4.1-4.2) in which we prove some uniform N bounds for two crucial probability integrals; in subsection (4.2) we prove the convergence of the invariant measures of (1) for $N \rightarrow +\infty$ towards a limit ρ . Then, we prove the tightness of the family of measures ν_N (subsection 4.3), the existence of a weak flow for equation (2) as an application of Skorokhod Theorem (subsection 4.4) and, eventually, we discussed the so-called invariance of the limit measure ρ under the weak flow in subsection 4.5.

2 Notations

In this section we fix up useful notations for the rest of the paper.

2.1 Spaces

Let $-2 \leq \sigma < -\frac{7}{4}$. Let $\varepsilon > 0$ and $\alpha \in]0, 1[$. Let $T \in \mathbb{R}$.

Given any variable x , we use the standard notations for $\langle x \rangle = \sqrt{1 + x^2}$ and $D_x = \sqrt{1 - \partial_x^2}$. We will denote with $S(\tau) = e^{i\tau\Delta}$.

For φ a non-negative increasing function, let \mathcal{X}_φ be the space induced by the norm

$$\|f\|_{\mathcal{X}_\varphi} = \|(1 + \varphi)^{-1} \langle x \rangle^{-6} D_x^\sigma f\|_{L^2}. \quad (3)$$

Even though this is not one of the spaces that we used in [12] to prove the invariance of some measure ρ under the flow of Schrödinger with a localised non linearity, one can prove that we have invariance in the topological σ algebra of this space for density reasons. We take the regularity to be less than two orders where one has invariance and the weights to be three times what they should be such that the derivative in time of the solution to $i\partial_t u = -\Delta u + |u|^2 u$ is in this space too. In view of what has been done in [8], this loss of derivative is maybe superfluous. The weight φ is needed as an artefact of the proof and might be unnecessary.

For convenience reasons, we introduce the space \mathcal{Z}_φ induced by the norm

$$\|f\|_{\mathcal{Z}_\varphi} = \|\langle x \rangle^{-2} (1 + \varphi)^{-1} D_x^{\sigma+2} f\|_{L^2} + \|\langle x \rangle^{-2} (1 + \varphi)^{-1/3} f\|_{L^6}. \quad (4)$$

Let $\mathcal{X}_\varphi^\varepsilon$ be the space induced by the norm

$$\|f\|_{\mathcal{X}_\varphi^\varepsilon} = \|(1 + \varphi)^{-1} \langle x \rangle^{-6(1+\varepsilon)} D_x^{\sigma(1+\varepsilon)} f\|_{L^2}. \quad (5)$$

We will prove later that the balls of \mathcal{X}_φ are compact in $\mathcal{X}_\varphi^\varepsilon$.

Let $\mathcal{X}_{T,\varphi}$ and $\mathcal{X}_{T,\varphi}^\varepsilon$ be the spaces defined as

$$\mathcal{X}_{T,\varphi} = C^\alpha([-T, T], \mathcal{X}_\varphi) \text{ and } \mathcal{X}_{T,\varphi}^\varepsilon = C([-T, T], \mathcal{X}_\varphi^\varepsilon) \quad (6)$$

where the index α is related to Lipschitz continuity in the sense that

$$\|f\|_{\mathcal{X}_{T,\varphi}} = \sup_{t_1, t_2 \in [-T, T]} \frac{\|f(t_1) - f(t_2)\|_{\mathcal{X}_\varphi}}{|t_1 - t_2|^\alpha} + \|f\|_{L^\infty([-T, T], \mathcal{X}_\varphi)}.$$

The idea is that the balls of $\mathcal{X}_{T,\varphi}$ are compact in $\mathcal{X}_{T,\varphi}^\varepsilon$.

Let \mathbf{m} be a measure and $p \in [1, \infty]$. By $L_{\mathbf{m}}^p$ we denote the space induced by the norm

$$\|F\|_{L_{\mathbf{m}}^p} = \left(\int |F(u)|^p d\mathbf{m}(u) \right)^{1/p}$$

if $p < \infty$ or

$$\|F\|_{L_{\mathbf{m}}^\infty} = \sup\{\lambda \geq 0 \mid \mathbf{m}(|F| \geq \lambda) \neq 0\}.$$

2.2 Measures

Let μ_N be the measure defined as

$$d\mu_N(u) = D_N^{-1} e^{-\frac{1}{2} \int_{-N}^N |u(x)|^4 dx} dq(u)$$

where q is the complex valued oscillator process given in the book by Simon [25]. We give more details about this process in Subsection 3.1, and D_N is the L_q^1 norm of $e^{-\frac{1}{2} \int_{-N}^N |u(x)|^4 dx}$. We remark that D_N goes to 0 when N goes to ∞ .

Let χ_N be C^∞ functions with compact supports such that for all $x \in \mathbb{R}$, $\chi_N(x) \in [0, 1]$,

$$\chi_N(x) = 1 \text{ on } [-N, N] \text{ and } \chi_N(x) = 0 \text{ outside } [-N - D_N^3, N + D_N^3]. \quad (7)$$

We call ρ_N the invariant measure defined in [12] under the flow of

$$i\partial_t u = -\Delta u + \chi_N |u|^2 u. \quad (8)$$

We call ψ_N the flow of this equation.

Let ν_N be the measure defined on the topological σ algebra of $\mathcal{X}_{T,\varphi}$ as for all A

$$\nu_N(A) = \rho_N(\{u_0 \mid t \mapsto \psi_N(t)u_0 \in A\}). \quad (9)$$

3 Previous results and corollary

3.1 Convergence in the whole line

We begin with the following definition.

Definition 3.1. A family $\{q(x)\}_{x \in \mathbb{R}}$ of Gaussian random variables is called an *oscillator process* or Ornstein-Uhlenbeck process if

$$\mathbb{E}(q(x)q(y)) = \frac{1}{2} e^{-|x-y|}.$$

We will denote with dq the measure on paths $\omega(x)$ associated to the oscillator process.

In analogy with what happens with Brownian motions, it is natural to link oscillator processes with some suitable semi group. We explain this connection in the following result.

Proposition 3.2. Let $L_0 = -\frac{1}{2} \frac{d^2}{dx^2} + \frac{1}{2} x^2 - \frac{1}{2}$ and $\Omega_0(x) = \pi^{-1/4} e^{-(1/2)x^2}$ so that $L_0 \Omega_0 = 0$ and $\int |\Omega_0|^2 = 1$. Let moreover $f_0, \dots, f_n \in L^\infty(\mathbb{R})$ and let $-\infty < y_0 < \dots < y_n < \infty$. Then

$$\mathbb{E}(f_0(q(y_0)), \dots, f_n(q(y_n))) = (\Omega_0, M_{f_0} e^{-x_1 L_0} M_{f_1} \dots e^{-x_n L_0} M_{f_n} \Omega_0)_{L^2}$$

where $x_i = y_i - y_{i-1} > 0$, $(\cdot, \cdot)_{L^2}$ denotes here the standard L^2 scalar product and M_f the multiplication operator $M_f g(x) = f(x)g(x)$.

Proof. See [25] Theorem 4.7 pag. 37. □

Remark 3.1. Proposition 3.2 yields an explicit kernel $Q_x(u_1, u_2)$ for the semi group e^{-xL_0} . Following the lines of the proof indeed we have

$$e^{-xL_0} f(u_1) = \int Q_x(u_1, u_2) f(u_2) du_2$$

where

$$Q_x(u_1, u_2) = \frac{1}{\sqrt{\pi(1 - e^{-2x})}} \exp \left(-\frac{\frac{1}{2}(u_1^2 + u_2^2)(1 + e^{-2x}) - 2e^{-x}u_1u_2}{1 - e^{-2x}} \right)$$

This is known as Mehler's formula.

We remark that Q_x is smooth and that for all $t \geq 0$ and $u_1 \neq u_2$

$$Q_x(u_1, u_2) \lesssim 1 + |u_1 - u_2|^{-1} \text{ and } |\partial_x Q_x(u_1, u_2)| \lesssim 1 + |u_1 - u_2|^{-3/2} |u_1 + u_2|. \quad (10)$$

We also have that $\partial_x Q(x = 0) = 0$ and

$$|\partial_x^2 Q_x(u_1, u_2)| \lesssim 1 + |u_1 - u_2|^{-5/2} |u_1 + u_2|^2. \quad (11)$$

And finally, we get that $\partial_x^2 Q_x(x = 0) = 0$ and

$$|\partial_x^3 Q_x(u_1, u_2)| \lesssim 1 + |u_1 - u_2|^{-7/2} |u_1 + u_2|^3. \quad (12)$$

Remark 3.2. Actually, minor modifications in the proof allow to adapt this result to higher dimensions: in this case the natural semi group will be indeed given by $L_0 = -\frac{1}{2}\Delta + \frac{1}{2}x^2 - \frac{1}{2}$.

The next step consists now in giving the analogue of Proposition 3.2 in a slightly more general setting, i.e. to relate the semi group e^{-xL} with $L = L_0 + V$ for some suitable potential V to path integrals. Results of this kind have been widely investigated in literature, especially in the case of Brownian motion, and are usually referred to as *Feynman-Kac formulas*. In what follows V will be any polynomial bounded from below, so that $E(V) = \inf \text{spec}(L_0 + V)$ is a simple eigenvalue with an associated strictly positive eigenvector Ω_V (some more general potentials can be considered, but we do not strive to cover the most general case here as discussed in [23]). We will denote with $\hat{L} = L_0 + V - E(V)$.

Definition 3.3. We define the $P(\phi)_1$ -process as the stochastic process with joint distribution of $q(x_1), \dots, q(x_n)$ ($x_1 < \dots < x_n$):

$$\Omega_V(u_1) \Omega_V(u_n) e^{-y_1 \hat{L}}(u_1, u_2) \dots e^{-y_{n-1} \hat{L}}(u_{n-1}, u_n)$$

where $e^{-y \hat{L}}(a, b)$ is the integral kernel of $e^{-s \hat{L}}$ and $y_i = x_{i+1} - x_i$. We will denote with $d\rho_V$ the corresponding measure.

Remark 3.3. In view of what will follow in the next section, it is important to give some estimate on the ground state $\Omega_V(x)$ (which is a regular function), with $V \cong |x|^4$. This can be done by means of the so called *WKB approximation scheme*, which gives the asymptotic behaviour $\Omega_{|x|^4}(x) \cong e^{-|x|^3}$ for $|x| \rightarrow +\infty$. We refer to [2], [27] for details.

Therefore, we are ready to state the following result.

Theorem 3.4. [Feynman-Kac] For any smooth and bounded test function $G : C(\mathbb{R}, \mathbb{C}) \rightarrow \mathbb{R}$, we have

$$\int G(u) d\rho_V(u) = \lim_{N \rightarrow +\infty} D_N^{-1} \int G(q) e^{-\int_{-N}^N V(q(s)) ds} dq$$

where D_N is the $L^1(dq)$ norm of $e^{-\int_{-N}^N V(q(s)) ds}$

Proof. See [25] Theorems 6.7 and 6.9 pag 58. Notice that, by mimicking the proof of Theorem 6.1 there, it is possible to deal also with the complex case. \square

Remark 3.4. This Theorem implies in particular the convergence of the sequence μ_N as defined in the introduction: the choice of the potential $V = |x|^4$ falls indeed within the assumptions we made for Definition 3.3 and therefore for applying Feynman-Kac Theorem. In what follows, we will omit the dependence on $V = |x|^4$ for the limit measure simply denoting it with ρ .

The reason for introducing all this framework is in the following result, in which we show that the Gaussian part of the invariant measure for NLS built in [12] is a complex valued oscillator process in the sense of the next proposition.

Proposition 3.5. Suppose $W_n(\omega)$ is the reunion of two complex, independent Wiener processes in n , $W_n^1(\omega)$, $W_n^2(\omega)$ and let

$$\phi(x) = \frac{1}{2\pi} \int_{\mathbb{R}} \frac{e^{inx}}{\sqrt{1+n^2}} dW_n(\omega) \quad (13)$$

where $x \in \mathbb{R}$ and $\omega \in \Omega$. Then it is possible to decompose

$$\phi(x) = \phi_1(x) + i\phi_2(x)$$

where ϕ_1, ϕ_2 are real-valued and independent. Moreover, each $\phi_j(x)$ is an oscillator process, as in Definition 3.1.

Remark 3.5. Before we prove this proposition, we remark that the process W_n is a random Gaussian field, see [24], such that $W_0 = 0$ and

$$\mathbb{E}(\overline{dW_{n_1}} dW_{n_2}) = dn_1 \delta(n_1 - n_2)$$

or equivalently

$$\mathbb{E}(\overline{W_{n_1}} W_{n_2}) = \begin{cases} 0 & \text{if } n_1 n_2 < 0 \\ \min(|n_1|, |n_2|) & \text{otherwise.} \end{cases}$$

Proof. In view of our assumption on $W_n(\omega)$, we have

$$\begin{aligned} \phi(x) &= \int_{\mathbb{R}} \frac{\cos(nx) dW_n^1(\omega) - \sin(nx) dW_n^2(n)}{\sqrt{1+n^2}} \\ &\quad + i \int_{\mathbb{R}} \frac{\sin(nx) dW_n^1(\omega) + \cos(nx) dW_n^2(n)}{\sqrt{1+n^2}} \\ &= \phi_1(x) + i\phi_2(x). \end{aligned}$$

To prove the independence, we rely on the Ito isometry to write

$$\begin{aligned}\mathbb{E}(\phi_1(x)\phi_2(y)) &= \int_{\mathbb{R}} \frac{\cos(nx) \sin(ny)}{1+n^2} dn - \int_{\mathbb{R}} \frac{\cos(nx) \sin(ny)}{1+n^2} dn \\ &= \int_{\mathbb{R}} \frac{\sin(n(x-y))}{1+n^2} dn \\ &= 0.\end{aligned}$$

Since ϕ_1, ϕ_2 are centred Gaussian variables, this implies that ϕ_1 and ϕ_2 are independent.

We now come to the second part of the proposition. First of all, we observe that

$$\begin{aligned}\mathbb{E}(\phi_1(x)\phi_1(y)) &= \mathbb{E}(\phi_2(x)\phi_2(y)) \\ &= \int_{\mathbb{R}} \frac{\cos(nx) \cos(ny) + \sin(nx) \sin(ny)}{1+n^2} dn \\ &= \int_{\mathbb{R}} \frac{\cos(n(x-y))}{1+n^2} dn.\end{aligned}\tag{14}$$

Let us then consider the function $F(x) = \int_{\mathbb{R}} \frac{\cos(nx)}{1+n^2} dn$: we aim to prove that

$$F(x) = \alpha \sinh(|x|) + \beta \cosh(x).\tag{15}$$

Let $\psi \in C_c^\infty$ be a test function. We have by considering Fourier transform

$$\begin{aligned}\langle F, (1 - \partial_x^2)\psi \rangle &= \left\langle \int_{\mathbb{R}} \cos(nx), \psi \right\rangle \\ &= \operatorname{Re} \left[\int_{\mathbb{R}} dx \psi(x) \int_{\mathbb{R}} e^{inx} dn \right] \\ &= \operatorname{Re} \left[\int_{\mathbb{R}} dn \int_{\mathbb{R}} e^{inx} \psi(x) dx \right] \\ &= \sqrt{2\pi} \operatorname{Re} \left[\int_{\mathbb{R}} \hat{\psi}(n) dn \right] \\ &= 2\pi \psi(0).\end{aligned}$$

On the other hand,

$$\begin{aligned}\langle \sinh(|x|), (1 - \partial_x^2)\psi \rangle &= \int_0^{+\infty} \sinh(x)(1 - \partial_x^2)\psi - \int_{-\infty}^0 \sinh(x)(1 - \partial_x^2)\psi. \\ &= I + II.\end{aligned}$$

The first integral gives

$$I = \int_0^{+\infty} \sinh(x)\psi - \int_0^{+\infty} \sinh(x)\partial_x^2\psi = I_1 - I_2$$

where integrating by parts

$$\begin{aligned}I_2 &= \sinh(x)\partial_x\psi \Big|_0^{+\infty} - \int_0^{+\infty} \cosh(x)\partial_x\psi \\ &= -\cosh(x)\psi \Big|_0^{+\infty} + \int_0^{+\infty} \sinh(x)\psi \\ &= \psi(0) + I_1\end{aligned}$$

and thus $I = -\psi(0)$.

Analogously,

$$\begin{aligned} -II &= \int_{-\infty}^0 \sinh(x)(1 - \partial_x^2)\psi \\ &= II_1 - II_2, \end{aligned}$$

and

$$\begin{aligned} II_2 &= \int_{-\infty}^0 \sinh(x)\partial_x^2\psi \\ &= \sinh(x)\partial_x\psi|_{-\infty}^0 - \int_{-\infty}^0 \cosh(x)\partial_x\psi \\ &= -\cosh(x)\psi|_{-\infty}^0 + \int_{-\infty}^0 \sinh(x)\psi \\ &= -\psi(0) + II_1 \end{aligned}$$

which implies $-II = \psi(0)$. Therefore, we have showed that

$$\langle \sinh(|x|), (1 - \partial_x^2)\psi \rangle = -2\psi(0).$$

On the other hand,

$$\langle \cosh(x), (1 - \partial_x^2)\psi \rangle = \langle (1 - \partial_x^2)\cosh(x), \psi \rangle = 0.$$

Putting all together, we thus have

$$F(x) = -\pi(\sinh(|x|) - \cosh(x)),$$

as $F(0) = \pi$. Hence, for $x \geq 0$, we have

$$F(x) = \pi \left(\frac{e^x + e^{-x} - e^x + e^{-x}}{2} \right) = \pi e^{-x}.$$

Getting back to (14) this gives, when $x \geq y$,

$$\mathbb{E}(\phi_j(x)\phi_j(y)) = \pi e^{-(x-y)}, \quad j = 1, 2$$

and this concludes the proof. □

Remark 3.6. We can also remark that $e^{-|x-y|}$ is the Green function of the operator $(1 - \partial_x^2)^{-1}$.

As a concluding result for this subsection, we give the following Proposition which is just a consequence of what we have seen so far.

Proposition 3.6. Let ρ_N be the invariant measure defined in [12]. Then

$$d\rho_N(u) = \frac{1}{D'_N} e^{-\int \chi_N |u(x)|^4 dx} dq(u). \quad (16)$$

where D'_N is the $L^1(dq)$ norm of $e^{-\int \chi_N |u(x)|^4 dx}$.

Additional remarks on where ρ is supported We wish to prove that ρ is supported by functions which are not localised, in particular, in the sense that they are not L^2 . For this, we remark that, thanks to the description of ρ as the $P(\phi)_1$ process given by Definition 3.3, we have that ρ is invariant under translations, that is for all test functions F and all $x_0 \in \mathbb{R}$:

$$\int F(f) d\rho(f) = \int F(f_{x_0}) d\rho(f)$$

with $f_{x_0}(x) = f(x - x_0)$. Besides, f is ρ -almost surely continuous and the law of $f(x)$ is absolutely continuous with regard to the Lebesgue measure and with density Ω_V^2 , that is

$$\rho(f(x) \in [f, f + df]) = \Omega_V^2(f) df.$$

Proposition 3.7. *The measure ρ is supported by non localised functions in the sense that ρ almost surely $f(x)$ does not go to 0 when x goes to ∞ .*

Proof. We compute the probability such that $f(x)$ goes to 0 when x goes to ∞ . We have

$$\rho(f(x) \rightarrow 0) = \rho(\forall \varepsilon > 0 \exists R \text{ such that } \forall x \geq R, |f(x)| \leq \varepsilon).$$

Writing everything in terms of sets, we have

$$\rho(f(x) \rightarrow 0) = \rho\left(\bigcap_{\varepsilon > 0} \bigcup_{R \in \mathbb{R}} \bigcap_{x \geq R} (|f(x)| \leq \varepsilon)\right).$$

Because of decreasing continuity of ρ , we have

$$\rho(f(x) \rightarrow 0) = \inf_{\varepsilon} \rho\left(\bigcup_{R \in \mathbb{R}} \bigcap_{x \geq R} (|f(x)| \leq \varepsilon)\right).$$

Writing $\bigcup_{R \in \mathbb{R}} \bigcap_{x \geq R}$ as a \liminf , we get

$$\rho(f(x) \rightarrow 0) = \inf_{\varepsilon} \rho(\liminf_{x \rightarrow \infty} (|f(x)| \leq \varepsilon)).$$

We use Fatou's lemma to get

$$\rho(f(x) \rightarrow 0) \leq \inf_{\varepsilon} \liminf_{x \rightarrow \infty} \rho(|f(x)| \leq \varepsilon)$$

and the invariance of ρ under translations to get $\rho(|f(x)| \leq \varepsilon) = \rho(|f(0)| \leq \varepsilon)$ and thus

$$\rho(f(x) \rightarrow 0) \leq \inf_{\varepsilon} \rho(|f(0)| \leq \varepsilon).$$

Finally, we use again the decreasing continuity of ρ to get

$$\rho(f(x) \rightarrow 0) \leq \rho(f(0) = 0).$$

The law of $f(0)$ being absolutely continuous with regard to the Lebesgue measure, we have that

$$\rho(f(x) \rightarrow 0) = 0.$$

□

Proposition 3.8. *The measure ρ is supported by non localised functions in the sense that ρ almost surely f does not belong to L^2 .*

Proof. Let $R \in \mathbb{Z}$ and let

$$\|f\|_R^2 = \int_R^{R+1} |f(x)|^2 dx.$$

If f belongs to L^2 then the series of general term $\|f\|_R^2$ converges and hence $\|f\|_R$ goes to 0 when R goes to ∞ . Therefore

$$\rho(\|f\|_{L^2} < \infty) \leq \rho(\|f\|_R \rightarrow 0).$$

For the same reasons as in the proof of Proposition 3.7, we have

$$\rho(\|f\|_{L^2} < \infty) \leq \rho(\|f\|_0 = 0).$$

Since f is ρ almost surely continuous, we get that $\|f\|_0 = 0$ almost surely implies $f(0) = 0$ and thus

$$\rho(\|f\|_{L^2} < \infty) \leq \rho(f(0) = 0) = 0.$$

□

3.2 Prokhorov's theorem

In this section we present a classical result of probability theory, known as Prokhorov Theorem, that represents a crucial tool in our convergence argument, and essentially connects the concepts of weak compactness and tightness. We refer to [18], [19] for all the details and deeper insight on the topic. First of all, we recall the following

Definition 3.9 (Weak compactness). Let S be a metric space. A family $(\mathbf{m}_N)_{N \geq 1}$ of probability measures on the Borel σ -algebra $\mathcal{B}(S)$ is said to be *weakly compact* if from any sequence \mathbf{m}_N , $N = 1, 2, \dots$ of measures from the family one can extract a weakly convergent subsequence \mathbf{m}_{N_k} , $k = 1, 2, \dots$ that is $\mathbf{m}_{N_k} \rightarrow \mathbf{m}$ for some probability measure \mathbf{m} .

Remark 3.7. Note that the definition does not require $\mathbf{m} \in (\mathbf{m}_N)_N$.

Remark 3.8. We recall that weak convergence means that for all $F : S \rightarrow \mathbb{R}$ Lipschitz continuous and bounded we have

$$\mathbb{E}_{\mathbf{m}_{N_k}}(F) \rightarrow \mathbb{E}_{\mathbf{m}}(F).$$

The convergence in law is stronger as it means that for all $F : S \rightarrow \mathbb{R}$ bounded we have

$$\mathbb{E}_{\mathbf{m}_{N_k}}(F) \rightarrow \mathbb{E}_{\mathbf{m}}(F).$$

Definition 3.10 (Tightness). Let S be a metric space and $(\mathbf{m}_N)_{N \geq 1}$ a family of probability measures on the Borel σ -algebra $\mathcal{B}(S)$. The family $(\mathbf{m}_N)_N$ is said to be *tight* if for any $\varepsilon > 0$ it is possible to find a compact set $K_\varepsilon \subset S$ such that for all $N > 1$, $\mathbf{m}_N(K_\varepsilon) \geq 1 - \varepsilon$.

Theorem 3.11 (Prokhorov Theorem). If a family $(\mathbf{m}_N)_{N \geq 1}$ of probability measures on a metric space S is tight, then it is weakly compact. Moreover, on a separable complete metric space the two notions are equivalent.

Proof. See e.g. [19], pag 114.

□

Let us now explain how we will make use of this Theorem. We have already introduced the measure ν_N defined on the topological σ -algebra of $\mathcal{X}_{T,\varphi}^\varepsilon$ as the image measure by the map

$$\begin{aligned}\mathcal{X}_\varphi &\rightarrow \mathcal{X}_{T,\varphi} \\ \nu &\rightarrow (t \mapsto \psi_N(t)(\nu));\end{aligned}$$

notice that in particular, for any measurable function $F : \mathcal{X}_{T,\varphi} \rightarrow \mathbb{R}$,

$$\int_{\mathcal{X}_{T,\varphi}} F(u) d\nu_N = \int_{\mathcal{X}_\varphi} F(\psi_N(t)(\nu)) d\rho_N. \quad (17)$$

The idea now is to show that the sequence of measures $\{\nu_N\}_N$ is tight in the space $\mathcal{X}_{T,\varphi}^\varepsilon$ for any $\varepsilon > 0$ (this will be done in details in subsection 4.3). Therefore, the application of Theorem 3.11 yields the weak convergence (up to a subsequence) of $\{\nu_N\}_N$ towards a measure ν on $\mathcal{X}_{T,\varphi}^\varepsilon$.

3.3 Skorokhod's theorem

In this subsection, we give and comment Skorokhod's theorem and explain how we use it to get the existence of a weak solution to the cubic defocusing Schrödinger equation (2).

Theorem 3.12 (Skorokhod). *Let S be a metric space and let $(\mathbf{m}_N)_N$ be a sequence of measures on S converging weakly towards a measure \mathbf{m} on S . We assume that the supports of \mathbf{m}_N and \mathbf{m} are separable. Then, there exists a probability space and a sequence of random variables $(X_N)_N$ and a random variable X on this probability space such that*

- *the law of X_N is \mathbf{m}_N ,*
- *the law of X is \mathbf{m} ,*
- *the sequence $(X_N)_N$ converges almost surely towards X .*

We refer to [18] for the proof and some applications.

Let us now give some remarks. First, we have that the space $\mathcal{X}_{\varphi,T}^\varepsilon$, given by (6) is separable. Assuming that we have proven that the sequence of measures ν_N converges weakly, which we deduce in Subsection 4.3 from Prokhorov's theorem, we get the existence of a sequence of random variables X_N of law ν_N which converges towards X of law ν the limit of $(\nu_N)_N$ up to a subsequence.

We now explain why X_N can be written $X_N(t, x) = \psi_N(t)(Y_N)(x)$ such that $Y_N(x) = X_N(0, x)$ and its law is ρ_N .

Proposition 3.13. *Assume that X_N is a random variable on a probability space (Ω, \mathcal{F}, P) with value in $\mathcal{X}_{\varphi,T}^\varepsilon$ of law ν_N . Let $Y_N = X_N(t=0)$. Then, P -almost surely we have $X_N(t) = \psi_N(t)Y_N$ and the law of Y_N is ρ_N .*

Proof. Let A be the set

$$A = \{\omega \in \Omega \mid \forall t \in \mathbb{R}, X_N(t)(\omega) = \psi_N(t)X_N(t=0)(\omega)\}.$$

We can rewrite A as

$$A = \{\omega \in \Omega \mid \exists u_0 \in \mathcal{X}_\varphi^\varepsilon \forall t \in \mathbb{R}, X_N(t)(\omega) = \psi_N(t)u_0\}.$$

Indeed, if $\omega \in A$ then there exists $u_0 = X_N(t=0)(\omega)$ in $\mathcal{X}_\varphi^\varepsilon$ such that $X_N(t)(\omega) = \psi_N(t)u_0$. Conversely, if there exists $u_0 \in \mathcal{X}_\varphi^\varepsilon$ such that $X_N(t)(\omega) = \psi_N(t)u_0$ then $X_N(0)(\omega) = \psi_N(0)u_0 = u_0$.

Hence, as the law of X_N is ν_N ,

$$P(A) = \nu_N(\{u \mid \exists u_0 \in \mathcal{X}_\varphi^\varepsilon u(t) = \psi_N(t)u_0\}) = \nu_N(\psi_N(t)(\mathcal{X}_\varphi^\varepsilon)).$$

And we recall from the definition of ν_N that

$$\nu_N(B) = \rho_N(u_0 \mid \psi_N(t)u_0 \in B).$$

Therefore

$$P(A) = \rho_N(u_0 \mid \psi_N(t)u_0 \in \psi_N(t)(\mathcal{X}_\varphi^\varepsilon)) = \rho_N(\mathcal{X}_\varphi^\varepsilon) = 1.$$

In other terms, P almost surely $X_N(t)(\omega) = \psi_N(t)X_N(0)(\omega)$.

Let us prove that the law of $Y_N = X_N(t=0)$ is ρ_N . Let A be a measurable set of $\mathcal{X}_\varphi^\varepsilon$. We have

$$P(Y_N \in A) = P(X_N(t=0) \in A).$$

Since the law of X_N is ν_N , we get

$$P(Y_N \in A) = \nu_N(u \mid u(t=0) \in A).$$

And given the definition of ν_N we have

$$P(Y_N \in A) = \rho_N(\{u_0 \mid \psi_N(t)u_0 \in \{u \mid u(t=0) \in A\}\}) = \rho_N(A).$$

Hence the law of Y_N is ρ_N . □

Proposition 3.14. *Under the assumptions of Proposition 3.13, for all $t \in \mathbb{R}$ the law of $X_N(t)$ is ρ_N .*

Proof. As $X_N(t) = \psi_N(t)Y_N$, we have that the law of $X_N(t)$ is the image measure of ρ_N under $\psi_N(t)$ but since ρ_N is invariant under ψ_N , we get that the law of $X_N(t)$ is ρ_N . □

The idea is now that as $X_N(t) = \Psi_N(t)Y_N$, the random variable X is a weak solution of the cubic non linear Schrödinger equation (2), on the support of the limit measure ρ , see Subsection 4.4.

3.4 Invariance of ρ_N

In this subsection, we recall the result of [12], and explain the density argument which makes ρ_N invariant under ψ_N in \mathcal{Z}_φ .

In [12], we proved that the measures ρ_N were invariant under the flow ψ_N for some topology \mathcal{Y}_s induced by the norm

$$p_s(f) = \|\langle t \rangle^{-2} \langle x \rangle^{-2} D^s S(t)f\|_{L^2(t \in \mathbb{R}, x \in \mathbb{R})}$$

for $s < -1/2$. Indeed, as χ_N is C^∞ with compact support, it satisfies the hypothesis of Subsection 1.1 in [12]. This means that for all measurable bounded function F of \mathcal{Y}_s and all times $t \in \mathbb{R}$, we have

$$\mathbb{E}_{\rho_N}(F \circ \psi_N(t)) = \mathbb{E}_{\rho_N}(F).$$

We recall that $S(t) = e^{-it\Delta}$.

We wish to prove that this property is also true in \mathcal{Z}_φ . Namely, that for all measurable bounded function F of \mathcal{Z}_φ ,

$$\mathbb{E}_{\rho_N}(F \circ \psi_N(t)) = \mathbb{E}_{\rho_N}(F).$$

For this, we need the following lemmas.

Lemma 3.15. *For all non negative and increasing function φ , and for ρ_N almost all u , we have $\psi_N(t)u \in \mathcal{Z}_\varphi$.*

Proof. This is a consequence of Proposition 4.4, bullet 3 in [12]. Indeed, with a control of $\psi_N(t_n)u$ at discrete well-chosen times, one can apply the contraction argument for the well-posedness and deduce that

$$\Psi_N(t)u = \psi_N(t)u - S(t)u$$

belongs to H^{s_∞} with s_∞ given in Subsection 1.1 of [12] for all $t \in [t_n, t_{n+1}]$ and ultimately at all time. Given that s_∞ can be chosen as close as but strictly less than $1/2$ and that H^{s_∞} is embedded in L^6 if $s_\infty \geq 1/3$ and hence that H^{s_∞} is embedded in \mathcal{Z}_φ if $s_\infty \geq \max(1/3, 2 + \sigma)$, we get that $\Psi_N(t)u$ belongs ρ_N almost surely to \mathcal{Z}_φ .

The fact that $S(t)u$ belongs ρ_N almost surely to \mathcal{Z}_φ is a consequence of Proposition 2.5 in [12]. \square

Lemma 3.16. *Let w be a C^∞ function of \mathbb{R} with compact support. There exists a constant depending on w , $C(w)$, such that for all $u \in \mathcal{Y}_s$,*

$$\|w * u\|_{\mathcal{Z}_\varphi} \leq p_s(u).$$

Proof. We write $\mathcal{Z}_\varphi = \mathcal{Z}_\varphi^2 \cap \mathcal{Z}_\varphi^6$ where \mathcal{Z}_φ^2 is the space induced by the norm

$$\|\langle x \rangle^{-2}(1 + \varphi)^{-1} D_x^{\sigma+2} f\|_{L^2}$$

and \mathcal{Z}_φ^6 is the space induced by the norm

$$\|\langle x \rangle^{-2}(1 + \varphi)^{-1/3} f\|_{L^6}.$$

We proceed by duality. Let g be in the dual of \mathcal{Z}_φ^2 , that is

$$\|(1 + \varphi)\langle x \rangle^2 D^{-2-\sigma} g\|_{L^2} < \infty.$$

We estimate $\langle g, w * u \rangle$ where $\langle \cdot, \cdot \rangle$ is the inner product. We have $\langle g, w * u \rangle = \langle w_1 * g, u \rangle$ with $w_1(x) = w(-x)$. For all $t \in \mathbb{R}$, we have

$$\langle g, w * u \rangle = \langle \langle x \rangle^2 D^{-s} S(t)(w_1 * g), \langle x \rangle^{-2} D^s S(t)u \rangle.$$

And hence we get for all t ,

$$|\langle g, w * u \rangle| \leq \|\langle x \rangle^2 D^{-s} S(t)(w_1 * g)\|_{L_x^2} \|\langle x \rangle^{-2} D^s S(t)u\|_{L_x^2}$$

and as the left hand side does not depend on t , we can take the L^2 norm in time between 0 and 1 to get

$$|\langle g, w * u \rangle| \leq \sup_{t \in [0,1]} \left(\|\langle x \rangle^2 D^{-s} S(t)(w_1 * g)\|_{L_x^2} \right) \|\langle x \rangle^{-2} D^s S(t)u\|_{L_x^2} \|1\|_{L^2(t \in [0,1])}$$

which yields

$$|\langle g, w * u \rangle| \leq \sup_{t \in [0,1]} \|\langle x \rangle^2 D^{-s} S(t)(w_1 * g)\|_{L_x^2} p_s(u).$$

We estimate $\|\langle x \rangle^2 D^{-s} S(t)(w_1 * g)\|_{L_x^2}$. We consider the Fourier transform to get

$$\|\langle x \rangle^2 D^{-s} S(t)(w_1 * g)\|_{L_x^2} = \|D_k^2 \langle k \rangle^{-s} e^{-ik^2 t} \hat{w}_1(k) \hat{g}(k)\|_{L_x^2} = \|D_k^2 \langle k \rangle^{-s+2+\sigma} e^{-ik^2 t} \hat{w}_1(k) \langle k \rangle^{-2-\sigma} \hat{g}(k)\|_{L_x^2}.$$

We distribute D_k^2 to get

$$\begin{aligned} \|\langle x \rangle^2 D^{-s} S(t)(w_1 * g)\|_{L_x^2} &\leq \|D_k^2 \langle k \rangle^{-s+2+\sigma} e^{-ik^2 t} \hat{w}_1(k)\|_{L_k^\infty} \|\langle k \rangle^{-2-\sigma} \hat{g}(k)\|_{L_x^2} + \\ &\quad \|\langle k \rangle^{-s+2+\sigma} e^{-ik^2 t} \hat{w}_1(k)\|_{L_k^\infty} \|D_k^2 \langle k \rangle^{-2-\sigma} \hat{g}(k)\|_{L_x^2}. \end{aligned}$$

We have

$$\|\langle k \rangle^{-2-\sigma} \hat{g}(k)\|_{L_x^2} = \|D^{-2-\sigma} g\|_{L^2} \leq \|(1 + \varphi) \langle x \rangle^2 D^{-2-\sigma} g\|_{L^2}$$

and

$$\|D_k^2 \langle k \rangle^{-2-\sigma} \hat{g}(k)\|_{L_x^2} = \|\langle x \rangle^2 D^{-2-\sigma} g\|_{L^2} \leq \|(1 + \varphi) \langle x \rangle^2 D^{-2-\sigma} g\|_{L^2}.$$

Regarding w , we see that for $t \in [0, 1]$,

$$\begin{aligned} \|D_k^2 \langle k \rangle^{-s+2+\sigma} e^{-ik^2 t} \hat{w}_1(k)\|_{L_k^\infty} &\leq \|\langle k \rangle^{-s+4+\sigma} \hat{w}_1(k)\|_{L_k^\infty} + \|\langle k \rangle^{-s+2+\sigma} D_k^2 \hat{w}_1\|_{L_k^\infty} \\ \|\langle k \rangle^{-s+2+\sigma} e^{-ik^2 t} \hat{w}_1(k)\|_{L_k^\infty} &\leq \|\langle k \rangle^{-s+2+\sigma} \hat{w}_1(k)\|_{L_k^\infty} \end{aligned}$$

and taking the inverse Fourier transform

$$\begin{aligned} \|\langle k \rangle^{-s+4+\sigma} \hat{w}_1(k)\|_{L_k^\infty} &\leq \|D_x^{-s+4+\sigma} w_1\|_{L_x^1} \\ \|\langle k \rangle^{-s+2+\sigma} D_k^2 \hat{w}_1\|_{L_k^\infty} &\leq \|D_x^{-s+2+\sigma} \langle x \rangle^2 w_1\|_{L_x^1} \\ \|\langle k \rangle^{-s+2+\sigma} \hat{w}_1(k)\|_{L_k^\infty} &\leq \|D_x^{-s+2+\sigma} w_1\|_{L_x^1}. \end{aligned}$$

Given that w_1 is C^∞ with compact support, all these quantities are finite and

$$|\langle g, w * u \rangle| \leq C(w) \|\langle x \rangle^2 D^{-2-\sigma} g\|_{L^2} p_s(u).$$

Therefore, as it is true for all g in the dual of \mathcal{Z}_φ^2 ,

$$\|w * u\|_{\mathcal{Z}_\varphi^2} \leq C(w) p_s(u).$$

The same proof applies for \mathcal{Z}_φ^6 . □

Proposition 3.17. *The measure ρ_N is invariant under the flow ψ_N for the topological σ -algebra of \mathcal{Z}_φ .*

Proof. Let F be a bounded measurable function on \mathcal{Z}_φ . As for ρ_N almost all u , $\psi_N(t)u$ belongs to \mathcal{Z}_φ (Lemma 3.15), and since ρ_N is defined on \mathcal{Z}_φ , we have that $\mathbb{E}_{\rho_N}(F \circ \psi_N(t))$ is well-defined.

Let w_k be a sequence of C^∞ functions with compact supports which converges towards a Dirac delta. Let $F_k : u \mapsto F(w_k * u)$. Thanks to Lemma 3.16, we have that $u \mapsto w_k * u$ is continuous and hence measurable from \mathcal{Y}_s to \mathcal{Z}_φ and thus F_k is measurable and bounded on \mathcal{Y}_s . We deduce

$$\mathbb{E}_{\rho_N}(F_k \circ \psi_N(t)) = \mathbb{E}_{\rho_N}(F_k).$$

As $\psi_N(t)u$ belongs almost surely to \mathcal{Z}_φ and F is bounded, we can apply the dominated convergence theorem to pass to the limit when $k \rightarrow \infty$, which yields

$$\mathbb{E}_{\rho_N}(F \circ \psi_N(t)) = \mathbb{E}_{\rho_N}(F)$$

for all t and concludes the proof. □

4 Proof of the theorem

Before we start applying the results of the last section to prove the theorem, we state two useful and central lemmas.

4.1 Two technical results

Lemma 4.1. *Let $r \geq 1$, there exist a non-negative, even and increasing on \mathbb{R}^+ function φ_r such that for all $x \in \mathbb{R}$ and all $N \in \mathbb{N}$, we have*

$$\mathbb{E}_{\mu_N}(|u(x)|^r) \leq \varphi_r(x), \quad \mathbb{E}_\rho(|u(x)|^r) \leq \varphi_r(x).$$

Remark 4.1. *This result may be seen as a consequence of an estimate on the ground state Ω_V of L , [14], or as a consequence of a Brascamp-Lieb inequality, as in [5].*

Proof. Let $x \in \mathbb{R}$ and let $N \geq |x|$. We apply Theorem 6.7 in [25] page 57 with

$$G(u) = |u(x)|^r \Omega_0(u(-N)) \Omega_V^{-1}(u(-N)) \Omega_0(u(N)) \Omega_V^{-1}(u(N)) e^{-2E(V)N}$$

where we recall that Ω_V is the eigenstate associated to the non-degenerate first eigenvalue $E(V)$ of $L = -\frac{1}{2} \Delta_u + |u|^2 + |u|^4 - \frac{1}{2}$ and Ω_0 is the eigenstate associated to the non-degenerate first eigenvalue 0 of $L_0 = -\frac{1}{2} \Delta_u + |u|^2 - \frac{1}{2}$. We get on the one hand

$$\int G(u) \Omega_V(u(-N)) \Omega_0^{-1}(u(-N)) \Omega_V(u(N)) \Omega_0^{-1}(u(N)) e^{2E(V)N} d\mu_N(u) = \int |u(x)|^r d\mu_N(u)$$

and on the other hand

$$\begin{aligned} \int G(u) \Omega_V(u(-N)) \Omega_0^{-1}(u(-N)) \Omega_V(u(N)) \Omega_0^{-1}(u(N)) e^{2E(V)N} d\mu_N(u) = \\ \int \tilde{G}(u_{-N}, u_x, u_N) \Omega_V(u_N) \Omega_V(u_{-N}) e^{-(x+N)\hat{L}}(u_{-N}, u_x) e^{-(N-x)\hat{L}}(u_x, u_N) du_{-N} du_x du_N \end{aligned}$$

with

$$\tilde{G}(u_{-N}, u_x, u_N) = |u_x|^r \Omega_0(u_{-N}) \Omega_V^{-1}(u_{-N}) \Omega_0(u_N) \Omega_V^{-1}(u_N) e^{-2E(V)N}.$$

We recall that $e^{-s\hat{L}}(u_1, u_2)$ is the fundamental solution to $\partial_s y = -\hat{L}y$, that is

$$y(s, u_2) = \int du_1 e^{-s\hat{L}}(u_1, u_2) y(0, u_1)$$

and $\hat{L} = L - E(V)$.

By simplifying the Ω_V we get

$$\int |u(x)|^r d\mu_N(u) = \int |u_x|^r \Omega_0(u_{-N}) \Omega_0(u_N) e^{-2E(V)N} e^{-(x+N)\hat{L}}(u_{-N}, u_x) e^{-(N-x)\hat{L}}(u_x, u_N) du_{-N} du_x du_N.$$

Let $\hat{L}_0 = L_0 - E(V)$, we have $\hat{L} - \hat{L}_0 = |x|^4$, thus by the maximum principle, we get $e^{-s\hat{L}}(u_1, u_2) \leq e^{-s\hat{L}_0}(u_1, u_2)$. Therefore

$$\begin{aligned} \int |u(x)|^r d\mu_N(u) \leq \\ \int |u_x|^r \Omega_0(u_{-N}) \Omega_0(u_N) e^{-2E(V)N} e^{-(x+N)\hat{L}_0}(u_{-N}, u_x) e^{-(N-x)\hat{L}_0}(u_x, u_N) du_{-N} du_x du_N. \end{aligned}$$

By definition of Ω_0 , we have that

$$\int du_1 \Omega_0(u_1) e^{-s\hat{L}_0}(u_1, u_2) = e^{sE(V)} \Omega_0(u_2) = \int du_1 \Omega_0(u_1) e^{-s\hat{L}_0}(u_2, u_1).$$

Hence integrating over u_N and u_{-N} yields

$$\int |u(x)|^r d\mu_N(u) \leq \int |u_x|^r \Omega_0(u_x)^2 e^{-2E(V)N} e^{(x+N)E(V)} e^{(N-x)E(V)} du_x$$

and thus

$$\int |u(x)|^r d\mu_N(u) \leq \int |u_x|^r \Omega_0(u_x)^2 du_x.$$

We have that $\Omega_0(u)$ behaves as $e^{-c|u|^2}$ hence the above quantity is finite. Therefore, there exists a constant, depending only on r , C_r such that for all $N \geq |x|$,

$$\int |u(x)|^r d\mu_N(u) \leq C_r.$$

Let

$$\varphi_r(x) = \max_{N < |x|} \mathbb{E}_{\mu_N}(|u(x)|^r) \leq C_r \max_{N < |x|} D_N^{-1} < \infty.$$

We have that φ_r is a non negative, increasing function.

Finally, from Theorem 6.9 in [25] page 58, we get

$$\mathbb{E}_\rho(|u(x)|^r) \leq C_r \leq \varphi_r(x).$$

□

We now include derivatives in our analysis.

Lemma 4.2. *Let $r \geq 2$ and $\alpha < \frac{1}{2}$ such that $0 \leq \alpha \leq \min(\frac{2}{r}, 1 - \frac{3}{2r})$, there exist a non-negative, increasing on \mathbb{R}^+ , even function $\varphi_{\alpha,r}$ such that for all $x, y \in \mathbb{R}$, $|x| \geq |y|$ and all $N \in \mathbb{N}$, we have*

$$\mathbb{E}_{\mu_N}\left(\frac{|u(x) - u(y)|^r}{|x - y|^{1+\alpha r}}\right) \leq \varphi_{\alpha,r}(x), \quad \mathbb{E}_\rho\left(\frac{|u(x) - u(y)|^r}{|x - y|^{1+\alpha r}}\right) \leq \varphi_{\alpha,r}(x).$$

Proof. We essentially use the same method as previously. Let $x, y \in \mathbb{R}$ and let $N \geq \max(|x|, |y|)$. We assume, without loss of generality, $x \geq y$. We apply Theorem 6.7 in [25] page 57 with

$$G(u) = \frac{|u(x) - u(y)|^r}{|x - y|^{1+\alpha r}} \Omega_0(u(-N)) \Omega_V^{-1}(u(-N)) \Omega_0(u(N)) \Omega_V^{-1}(u(N)) e^{-2E(V)N}$$

We get on the one hand

$$\int G(u) \Omega_V(u(-N)) \Omega_0^{-1}(u(-N)) \Omega_V(u(N)) \Omega_0^{-1}(u(N)) e^{2E(V)N} d\mu_N(u) = \int \frac{|u(x) - u(y)|^r}{|x - y|^{\alpha r + 1}} d\mu_N(u)$$

and on the other hand

$$\begin{aligned} \int G(u) \Omega_V(u(-N)) \Omega_0^{-1}(u(-N)) \Omega_V(u(N)) \Omega_0^{-1}(u(N)) e^{2E(V)N} d\mu_N(u) = \\ \int \frac{|u_x - u_y|^r}{|x - y|^{\alpha r + 1}} \Omega_0(u_{-N}) \Omega_V^{-1}(u_{-N}) \Omega_0(u_N) \Omega_V^{-1}(u_N) e^{-2E(V)N} \\ \Omega_V(u_N) \Omega_V(u_{-N}) e^{-(y+N)\hat{L}}(u_{-N}, u_y) e^{-(x-y)\hat{L}}(u_y, u_x) e^{-(N-x)\hat{L}}(u_x, u_N) du_{-N} du_y du_x du_N \end{aligned}$$

By simplifying the Ω_V we get

$$\int \frac{|u(x) - u(y)|^r}{|x - y|^{\alpha r + 1}} d\mu_N(u) = \int \tilde{G} e^{-(y+N)\hat{L}}(u_{-N}, u_y) e^{-(x-y)\hat{L}}(u_y, u_x) e^{-(N-x)\hat{L}}(u_x, u_N) du_{-N} du_y du_x du_N$$

with

$$\tilde{G}(u_{-N}, u_y, u_x, u_N) = \frac{|u_x - u_y|^r}{|x - y|^{\alpha r + 1}} \Omega_0(u_{-N}) \Omega_0(u_N) e^{-2E(V)N}.$$

Using as previously the maximum principle, we get

$$\int \frac{|u(x) - u(y)|^r}{|x - y|^{\alpha r + 1}} d\mu_N(u) \leq \int \frac{|u_x - u_y|^r}{|x - y|^{\alpha r + 1}} \Omega_0(u_{-N}) \Omega_0(u_N) e^{-2E(V)N} e^{-(y+N)\hat{L}_0}(u_{-N}, u_y) e^{-(x-y)\hat{L}_0}(u_y, u_x) e^{-(N-x)\hat{L}_0}(u_x, u_N) du_{-N} du_y du_x du_N.$$

Integrating over u_{-N} and u_N yields

$$\int \frac{|u(x) - u(y)|^r}{|x - y|^{\alpha r + 1}} d\mu_N(u) = \int \frac{|u_x - u_y|^r}{|x - y|^{\alpha r + 1}} \Omega_0(u_x) \Omega_0(u_y) e^{-(x-y)L_0}(u_y, u_x) du_y du_x.$$

We remark that the \hat{L}_0 as turned into L_0 as we simplified with $e^{-2E(V)N}$.

When $\alpha r \leq 1$, we use the estimates (10), (11) and the fact that the derivative at $z = 0$ of $e^{-zL_0}(u_1, u_2)$ is 0 outside the diagonal $u_1 = u_2$ to get

$$e^{-(x-y)L_0}(u_y, u_x) |x - y|^{-1-\alpha r} \lesssim (1 + |u_x - u_y|^{-3/2-\alpha r} |u_x + u_y|^{1+\alpha r})$$

We get

$$\int \frac{|u(x) - u(y)|^r}{|x - y|^{1+\alpha r}} d\mu_N(u) \leq C_{r,\alpha} \int |u_x - u_y|^{r-3/2-\alpha r} \Omega_0(u_x) \Omega_0(u_y) |u_x + u_y|^{\alpha r + 1} du_y du_x.$$

With the choice of α , $r - 3/2 - \alpha r$ is non-negative, and since $\Omega_0(u)$ behaves like $e^{-c|u|^2}$, the above quantity is finite and does not depend on x or y . Hence, there exists $C_{r,\alpha}$ such that for all $N \geq \max(|x|, |y|)$,

$$\int \frac{|u(x) - u(y)|^r}{|x - y|^{1+\alpha r}} d\mu_N(u) \leq C_{r,\alpha}.$$

When $1 \leq \alpha r \leq 2$, we use the estimates (11), (12) and the fact that the two first derivatives at $z = 0$ of $e^{-zL_0}(u_1, u_2)$ are 0 outside the diagonal $u_1 = u_2$ to get

$$e^{-(x-y)L_0}(u_y, u_x) |x - y|^{-1-\alpha r} \lesssim (1 + |u_x - u_y|^{-3/2-\alpha r} |u_x + u_y|^{1+\alpha r})$$

We get

$$\int \frac{|u(x) - u(y)|^r}{|x - y|^{1+\alpha r}} d\mu_N(u) \leq C_{r,\alpha} \int |u_x - u_y|^{r-3/2-\alpha r} \Omega_0(u_x) \Omega_0(u_y) |u_x + u_y|^{\alpha r + 1} du_y du_x.$$

With the choice of α , $r - 3/2 - \alpha r$ is non-negative, and since $\Omega_0(u)$ behaves like $e^{-c|u|^2}$, the above quantity is finite and does not depend on x or y . Hence, there exists $C_{r,\alpha}$ such that for all $N \geq \max(|x|, |y|)$,

$$\int \frac{|u(x) - u(y)|^r}{|x - y|^{1+\alpha r}} d\mu_N(u) \leq C_{r,\alpha}.$$

For $N \leq \max(|x|, |y|)$, we have

$$\int \frac{|u(x) - u(y)|^r}{|x - y|^{\alpha r + 1}} d\mu_N(u) \leq D_N^{-1} \mathbb{E} \left(\frac{|q(x) - q(y)|^r}{|x - y|^{\alpha r + 1}} \right).$$

As $\alpha < \frac{1}{2}$, we get that the mean value on the oscillator process is finite. Let

$$C'_{r,\alpha} = \left(\min \left(\mathbb{E} \left(\frac{|q(x) - q(y)|^r}{|x - y|^{\alpha r + 1}} \right), C_{r,\alpha} \right) \right)$$

and

$$\varphi_{\alpha,r}(x) = C'_{r,\alpha} \max_{N \leq |x|} D_N^{-1}.$$

For all N , $D_N \leq 1$, hence for all N ,

$$\mathbb{E}_{\mu_N} \left(\frac{|u(x) - u(y)|^r}{|x - y|^{\alpha r + 1}} \right) \leq \varphi_{\alpha,r}(\max(|x|, |y|)).$$

Finally, from Theorem 6.9 in [25] page 58, we get

$$\mathbb{E}_\rho \left(\frac{|u(x) - u(y)|^r}{|x - y|^{\alpha r + 1}} \right) \leq \varphi_{\alpha,r}(\max(|x|, |y|)).$$

□

4.2 Convergence of ρ_N towards ρ

In this subsection, we prove that the sequence ρ_N converges towards ρ in law.

Proposition 4.3. *For all non-negative increasing function φ , we have that the sequence ρ_N converges towards ρ in law in the sense that for all bounded measurable function F from $\mathcal{X}_\varphi^\varepsilon$ to \mathbb{R} , the sequence $\mathbb{E}_{\rho_N}(F)$ converges towards $\mathbb{E}_\rho(F)$.*

Lemma 4.4. *Let F be a measurable function from $\mathcal{X}_\varphi^\varepsilon$ to \mathbb{R} such that either F is bounded or there exists $x_0 \in \mathbb{R}$ and $r \geq 1$ satisfying $F(u) = |u(x_0)|^r$ or there exists x_0 and y_0 in \mathbb{R} , $r \geq 2$ and $\alpha \in [0, \frac{1}{2}[$ such that $\alpha \leq \frac{2}{r}$, $1 - \frac{3}{2r}$ satisfying $F(u) = \frac{|u(x_0) - u(y_0)|^r}{|x_0 - y_0|^{\alpha r + 1}}$. We have*

$$|\mathbb{E}_{\rho_N}(F) - \mathbb{E}_{\mu_N}(F)| \leq C_F \frac{D_N}{1 - D_N^2}$$

where C_F depends on F in the case F bounded and on r , or r and α , in the other cases but not on N .

Proof. Let $g_N(u) = e^{-\frac{1}{2} \int \chi_N(x) |u(x)|^4 dx}$, $h_N(x) = e^{-\frac{1}{2} \int_{-N}^N |u(x)|^4 dx}$ and set $D'_N = \int g_N(u) dq(u)$. We recall that $D_N = \int h_N(u) dq(u)$.

Given Proposition 3.6, we have

$$\mathbb{E}_{\rho_N}(F) = \int F(u) \frac{g_N(u)}{D'_N} dq(u).$$

By definition, we have

$$\mathbb{E}_{\mu_N}(F) = \int F(u) \frac{h_N(u)}{D_N} dq(u).$$

Let us prove that

$$\int |F(u)| |g_N(u) - h_N(u)| dq(u) \leq CD_N^3.$$

Indeed, as $\chi_N(x) \in [0, 1]$ and $\chi_N(x) = 1$ on $[-N, N]$, we have $g_N(u) \leq h_N(u)$ and thus

$$|g_N(u) - h_N(u)| \leq h_N(u) \frac{1}{2} \int |\chi_N(x) - 1_{[-N, N]}(x)| |u(x)|^4 dx.$$

Integrating with respect to $dq(u)$ and bounding $h_N(u)$ by 1 yields in the case that $F(u)$ is bounded

$$\int |F(u)| |g_N(u) - h_N(u)| dq(u) \leq C_F \int |\chi_N(x) - 1_{[-N, N]}(x)| \left(\int |u(x)|^4 dq(u) \right) dx.$$

In the cases $F(u) = |u(x_0)|^r$ or $F(u) = \frac{|u(x_0) - u(y_0)|^r}{|x_0 - y_0|^{1+\alpha r}}$, we get

$$\int |F(u)| |g_N(u) - h_N(u)| dq(u) \leq \int |\chi_N(x) - 1_{[-N, N]}(x)| \left(\int |F(u)| |u(x)|^4 dq(u) \right) dx.$$

Since dq is up to a constant the Gaussian law induced by the random variable

$$\int \frac{e^{inx}}{\sqrt{1+n^2}} dW(n)$$

where $W(n)$ is the reunion of two independent complex Brownian motions, we get

$$\int |u(x)|^4 dq(u) \leq C \left(\int |u(x)|^2 dq(u) \right)^2$$

where C is a universal constant related to Gaussian variables. We have $\int |u(x)|^2 dq(u) = \int \frac{dn}{1+n^2} = \pi$, thus

$$\int |u(x)|^4 dq(u) \leq C$$

where C does not depend on x .

We use the proof of Lemma 4.1 in the case $F(u) = |u(x_0)|^r$ and the proof of Lemma 4.2 in the case $F(u) = \frac{|u(x_0) - u(y_0)|^r}{|x_0 - y_0|^{1+\alpha r}}$ to get that

$$\int |F(u)| |u(x)|^4 dq(u)$$

is finite, depends on r or r and α but is independent from x and x_0 or x , x_0 , and y_0 .

We get

$$\int |F(u)| |g_N(u) - h_N(u)| dq(u) \leq C_F \frac{1}{2} \int |\chi_N(x) - 1_{[-N, N]}(x)| dx.$$

By definition of χ_N , it is equal to 1 on $[-N, N]$, to 0 outside $[-N - D_N^3, N + D_N^3]$ and belongs to $[0, 1]$, hence

$$\frac{1}{2} \int |\chi_N(x) - 1_{[-N, N]}(x)| \leq D_N^3$$

and

$$\int |F(u)| |g_N(u) - h_N(u)| dq(u) \leq C_F D_N^3.$$

We deduce from that $|D'_N - D_N| \leq D_N^3$ for $F = 1$ and thus $D'_N \geq D_N - D_N^3$.

We have

$$|\mathbb{E}_{\rho_N}(F) - \mathbb{E}_{\mu_N}(F)| \leq \int |F(u)| \frac{|g_N(u) - h_N(u)|}{D'_N} dq(u) + \int |F(u)| h_N(u) \frac{|D_N - D'_N|}{D_N D'_N}.$$

Given the previous estimates, we have

$$|\mathbb{E}_{\rho_N}(F) - \mathbb{E}_{\mu_N}(F)| \leq C_F \frac{D_N^3}{D'_N} + \int h_N(u) |F(u)| dq(u) \frac{D_N^3}{D_N D'_N}.$$

We get by bounding h_n by 1

$$|\mathbb{E}_{\rho_N}(F) - \mathbb{E}_{\mu_N}(F)| \leq C_F \frac{D_N^2}{D'_N} \leq C_F \frac{D_N}{1 - D_N^2}$$

which concludes the proof. \square

Proof of Proposition 4.3. As μ_N converges in law towards ρ for cylindrical sets, we get that μ_N converges towards ρ for the topological σ algebra of continuous functions u such that $\langle x \rangle^{-\nu} u$ belongs to L^∞ for any $\nu > 0$.

Indeed, let $\overline{B}(u_0, R)$ the closed ball of centre u_0 and radius R in this space, we have

$$\overline{B}(u_0, R) = \{u \in C(\mathbb{R}, \mathbb{C}) \mid \forall x \in \mathbb{R}, |u(x) - u_0(x)| \leq R \langle x \rangle^\nu\}.$$

As the u are continuous, we can restrict x to \mathbb{Q} and get

$$\overline{B}(u_0, R) = \bigcap_{x \in \mathbb{Q}} \{u \in C(\mathbb{R}, \mathbb{C}) \mid |u(x) - u_0(x)| \leq R \langle x \rangle^\nu\}.$$

As $\{u \in C(\mathbb{R}, \mathbb{C}) \mid |u(x) - u_0(x)| \leq R \langle x \rangle^\nu\}$ is a cylindrical set, we get the convergence of μ_N towards ρ for the balls of continuous functions in the norm $\|\langle x \rangle^{-\nu} \cdot\|_{L^\infty}$ and as these balls generates the topological σ algebra we get the convergence of μ_N towards ρ .

For this to be significant, we prove that

$$\rho(\{u \in C(\mathbb{R}, \mathbb{C}) \mid \|\langle x \rangle^{-\nu} u\|_{L^\infty} < \infty\}) = 1$$

and

$$\mu_N(\{u \in C(\mathbb{R}, \mathbb{C}) \mid \|\langle x \rangle^{-\nu} u\|_{L^\infty} < \infty\}) = 1.$$

Indeed, if we do not have these properties then the convergence in $\langle x \rangle^\nu L^\infty$ is only true on the set

$$\{u \in C(\mathbb{R}, \mathbb{C}) \mid \|\langle x \rangle^{-\nu} u\|_{L^\infty} < \infty\}$$

which has not a full ρ or μ_N measure and hence one cannot have the convergence in law.

Given that the μ_N are absolutely continuous with respect to q , it is enough to prove that u is dq almost surely continuous and such that $\langle x \rangle^{-\nu} u$ belongs to L^∞ .

Let $\alpha \in [0, \frac{1}{2}[$ and $p \in]1, \infty[$ such that $\frac{1}{p} < \min(\nu, \frac{1}{2} - \alpha)$, we have by Sobolev inequality, for $\frac{1}{p} < s < \frac{1}{2} - \alpha$,

$$\|\langle x \rangle^{-\nu} D^\alpha u\|_{L_q^p(L^\infty(\mathbb{R}))} \leq C \|D^s \langle x \rangle^{-\nu} D^\alpha u\|_{L_q^p(L^p(\mathbb{R}))}.$$

As differentiating $\langle x \rangle^{-\nu}$ only gains in powers of x and since we can reverse the order of integration we get,

$$\|\langle x \rangle^{-\nu} D^\alpha u\|_{L_q^p(L^\infty(\mathbb{R}))} \leq C \|\langle x \rangle^{-\nu} D^{s+\alpha} u\|_{L^p(\mathbb{R}, L_q^p)}.$$

Since $s + \alpha < \frac{1}{2}$ and dq is a Gaussian, we have

$$\|D^{s+\alpha}u(x)\|_{L_q^p} \leq C_p \|D^{s+\alpha}u(x)\|_{L^2(dq)} \leq C_p \left(\int \frac{dn}{(1+n^2)^{1-(s+\alpha)}} \right)^{1/2} < \infty.$$

And since $\langle x \rangle^{-\nu}$ belongs to L^p , we get

$$\|\langle x \rangle^{-\nu} D^\alpha u\|_{L_q^p L^\infty(\mathbb{R})} < \infty$$

which yields that $\|\langle x \rangle^{-\nu} D^\alpha u\|_{L^\infty(\mathbb{R})}$ is μ_N almost surely finite and hence $\langle x \rangle^{-\nu} u$ belongs μ_N almost surely to $W^{\alpha,\infty}$ which ensures that u is μ_N almost surely continuous and that $\langle x \rangle^{-\nu} u$ belongs μ_N almost surely to L^∞ .

For ρ , we use Theorem 6.9 in [25] to get that for $r > 4$ and $\alpha \in]\frac{1}{r}, \frac{2}{r}[$ (the couple (r, α) satisfies the assumption in Lemma 4.2), we have that

$$\mathbb{E}_\rho \left(\frac{|u(x) - u(y)|^r}{|x - y|^{1+\alpha r}} \right)$$

is bounded uniformly in x and y . Hence for $\nu > \frac{1}{r}$ we get that

$$\mathbb{E}_\rho \left(\int dx \int dy \langle x \rangle^{-\nu r} \langle y \rangle^{-\nu r} \frac{|u(x) - u(y)|^r}{|x - y|^{1+\alpha r}} \right)$$

is finite and hence $\langle x \rangle^{-\nu} u$ belongs ρ almost surely to $W^{\alpha,r}$ which ensures that u is ρ almost surely continuous and that $\langle x \rangle^{-\nu} u$ belongs ρ almost surely to L^∞ .

The topology of $\mathcal{X}_\varphi^\varepsilon$ is weaker than the topology of $\langle x \rangle^\nu L^\infty$. Indeed,

$$\|u\|_{\mathcal{X}_\varphi^\varepsilon} = \|(1 + \varphi)^{-1} \langle x \rangle^{-6(1+\varepsilon)} D^{\sigma(1+\varepsilon)} u\|_{L^2} \leq \|D^{-\sigma(1+\varepsilon)} \langle x \rangle^{-6(1+\varepsilon)} D^{\sigma(1+\varepsilon)} u\|_{L^2}.$$

We recall that $\sigma < 0$ and that differentiating $\langle x \rangle^{-6(1+\varepsilon)}$ only gains in powers of x thus

$$\|u\|_{\mathcal{X}_\varphi^\varepsilon} \leq C \|\langle x \rangle^{-6(1+\varepsilon)} u\|_{L^2}$$

and by Hölder inequality

$$\|u\|_{\mathcal{X}_\varphi^\varepsilon} \leq C \|\langle x \rangle^{-\nu} u\|_{L^\infty}.$$

Thus, we get that μ_N converges towards ρ in law for the topological σ algebra of $\mathcal{X}_\varphi^\varepsilon$. This implies that for all F measurable, bounded from $\mathcal{X}_\varphi^\varepsilon$ to \mathbb{R} , we have that $\mathbb{E}_{\mu_N}(F)$ converges towards $\mathbb{E}_\rho(F)$. Given the lemma, we get that $\mathbb{E}_{\rho_N}(F)$ converges towards $\mathbb{E}_\rho(F)$ which implies the result. \square

We deduce from Lemmas 4.1, 4.2 and 4.4 the following lemma.

Lemma 4.5. *Let $r \geq 2$ and $\alpha \in [0, \frac{1}{2}[$ such that $\alpha r \leq 2, 1 - \frac{3}{2r}$, there exists φ_r and $\varphi_{r,\alpha}$ two non negative increasing functions such that for all $N \in \mathbb{N}$ and all $x, y \in \mathbb{R}$, $|x| \geq |y|$ we have*

$$\mathbb{E}_{\rho_N}(|u(x)|^r) \leq \varphi_r(x) \tag{18}$$

and

$$\mathbb{E}_{\rho_N} \left(\frac{|u(x) - u(y)|^r}{|x - y|^{1+\alpha r}} \right) \leq \varphi_{r,\alpha}(x). \tag{19}$$

4.3 Tightness of ν_N

This section is devoted to prove the tightness of the family of measures $(\nu_N)_N$. In order to do this, we need to begin by proving some preliminary technical results. We begin with the following compactness argument.

Proposition 4.6. *Let $R_0 > 0$. The set $K = \{u : \|u\|_{X_\varphi} \leq R_0\}$ is compact in X_φ^ε .*

Proof. We show that for every $\varepsilon > 0$ there exists n_ε and $u_1, \dots, u_{n_\varepsilon}$ such that

$$K \subset \bigcup_{j=1}^{n_\varepsilon} B(u_j, \varepsilon)$$

where B are the balls in the X_φ^ε topology. To do that, we introduce a smooth cut-off function $1_{|x| \leq R}$ such that $1_{|x| \leq R}(x) = 1$ for $x \in [-R, R]$ and $1_{|x| \leq R}(x) = 0$ for $x \in (-\infty, -2R) \cup (2R, +\infty)$. We then have, for any $u \in K$,

$$\|u\|_{X_\varphi^\varepsilon} \leq I + II$$

where

$$I = \|(1 + \varphi)^{-1} \langle x \rangle^{-6(1+\varepsilon)} D^{\sigma(1+\varepsilon)} (1 - 1_{|x| \leq R}) u\|_{L^2}$$

and

$$II = \|(1 + \varphi)^{-1} \langle x \rangle^{-6(1+\varepsilon)} D^{\sigma(1+\varepsilon)} 1_{|x| \leq R} u\|_{L^2}$$

The first term is easily bounded as follows

$$I \leq CR^{-\varepsilon} \|(1 + \varphi)^{-1} \langle x \rangle^{-6} D^{\sigma(1+\varepsilon)} u\|_{L^2} \leq CR^{-\varepsilon} R_0. \quad (20)$$

To estimate the second term, we need to introduce also a frequency cut-off Π_N

$$\widehat{\Pi_N f}(n) = \eta\left(\frac{n}{N}\right) \hat{f}(n)$$

with η a non negative even C^∞ function with compact support included in $[-1, 1]$ and such that $\eta = 1$ on $[-1/2, 1/2]$ and $N > 0$. We thus rewrite

$$II = II_A + II_B$$

where

$$II_A = \|(1 + \varphi)^{-1} \langle x \rangle^{-6(1+\varepsilon)} D^{\sigma(1+\varepsilon)} (1 - \Pi_N) 1_{|x| \leq R} u\|_{L^2}$$

and

$$II_B = \|(1 + \varphi)^{-1} \langle x \rangle^{-6(1+\varepsilon)} D^{\sigma(1+\varepsilon)} \Pi_N 1_{|x| \leq R} u\|_{L^2}$$

To estimate II_A we use the fact that Π_N cuts off high frequencies, yielding

$$\begin{aligned} II_A &\leq C_R N^{-\varepsilon} \|(1 + \varphi)^{-1} \langle x \rangle^{-6} D^\sigma 1_{|x| \leq R} u\|_{L^2} \\ &\leq C_R N^{-\varepsilon} R_0. \end{aligned} \quad (21)$$

Finally, to estimate II_B we use that $\Pi_N 1_{|x| \leq R} u$ is finite dimensional, and therefore for every $\varepsilon > 0$ there exist n_ε and $u_1, \dots, u_{n_\varepsilon} \in \Pi_N 1_{|x| \leq R} K$ such that

$$\Pi_N 1_{|x| \leq R} K \subset \bigcup_{j=1}^{n_\varepsilon} B(u_j, \varepsilon/3). \quad (22)$$

We are now ready to conclude: for a fixed $\varepsilon > 0$, we can choose R in (20) big enough such that $I \leq \varepsilon/3$ and, for fixed ε and R , we can choose N in (21) big enough such that $II_A \leq \varepsilon/3$. Therefore, taking any $u \in K$, we can conclude that there exist $j \in \{1, \dots, n_\varepsilon\}$ such that taking the corresponding $u_j \in \mathcal{X}_\varepsilon$ in II_B gives

$$\|u - u_j\|_{\mathcal{X}_\varepsilon} \leq \frac{2}{3}\varepsilon + \|\Pi_N 1_{|x| \leq R} u - u_j\|_{\mathcal{X}_\varepsilon} \leq \varepsilon$$

and thus the proof is concluded. \square

As a consequence, we have the following

Corollary 4.7. *For every $\varepsilon > 0$ the embedding $\mathcal{X}_\varphi \subset \mathcal{X}_\varphi^\varepsilon$ is compact.*

Another crucial tool is represented by the following uniform estimates.

Proposition 4.8. *Let $0 < \alpha \leq 1/2$. Then there exists a non negative increasing function $\varphi(x)$ such that*

$$\|u\|_{L_{v_N}^2, \mathcal{X}_{T,\varphi}} \quad (23)$$

is uniformly bounded in N .

We go step by step and we start by explaining the reason why we introduced the space \mathcal{Z}_φ .

Lemma 4.9. *There exists a constant $C(T)$ independent from φ such that for all N*

$$\|u\|_{L_{v_N}^2, \mathcal{X}_{T,\varphi}} \leq C(T) \|u_0\|_{L_{p_N}^2, \mathcal{Z}_\varphi}.$$

Proof. The ideas of the proof are two fold : the first one is that we can estimate the α Lipschitz continuity by bounding $\partial_t u$ which we know explicitly in terms of u as u is v_N almost surely the solution to (8), the second one is that ρ_N is invariant under the flow of (8).

We recall the definition of the $\|\cdot\|_{\mathcal{X}_{T,\varphi}}$ given in the introduction to be

$$\|u\|_{\mathcal{X}_{T,\varphi}} = \sup_{t_1, t_2 \in [-T, T]} \frac{\|u(t_1) - u(t_2)\|_{\mathcal{X}_\varphi}}{|t_1 - t_2|^\alpha} + \|u\|_{L^\infty([-T, T], \mathcal{X}_\varphi)} \quad (24)$$

(we will fix later the weight function φ). We observe that, by Hölder inequality,

$$\begin{aligned} \|u(t_1) - u(t_2)\|_{\mathcal{X}_\varphi} &= \|(1 + \varphi)^{-1} \langle x \rangle^{-6} D_x^\sigma (u(t_1) - u(t_2))\|_{L^2} \\ &= \left\| (1 + \varphi)^{-1} \langle x \rangle^{-6} D_x^\sigma \int_{t_1}^{t_2} \partial_\tau u(\tau) d\tau \right\|_{L^2} \\ &\leq |t_1 - t_2|^{1/2} \|\partial_t u\|_{L^2([-T, T], \mathcal{X}_\varphi)} \end{aligned}$$

and thus for every $\alpha \in (0, \frac{1}{2}]$, we get

$$\sup_{t_1, t_2 \in [-T, T]} \frac{\|u(t_1) - u(t_2)\|_{\mathcal{X}_\varphi}}{|t_1 - t_2|^\alpha} \leq C(T) (\|\partial_t u\|_{L^2([-T, T], \mathcal{X}_\varphi)} + \|u\|_{L^2([-T, T], \mathcal{X}_\varphi)}).$$

By Sobolev embeddings, we have

$$\|u\|_{L^\infty([-T, T], \mathcal{X}_\varphi)} \leq C(T) (\|\partial_t u\|_{L^2([-T, T], \mathcal{X}_\varphi)} + \|u\|_{L^2([-T, T], \mathcal{X}_\varphi)}).$$

We take the $L_{\nu_N}^2$ norm to get

$$\|u\|_{L_{\nu_N}^2, L^\infty([-T, T], \mathcal{X}_\varphi)} \leq C(T)(\|\partial_t u\|_{L_{\nu_N}^2, L^2([-T, T], \mathcal{X}_\varphi)} + \|u\|_{L_{\nu_N}^2, L^2([-T, T], \mathcal{X}_\varphi)}).$$

We use the definition of ν_N as the image measure of ρ_N under $\psi_N(t)$ to get

$$\|u\|_{L_{\nu_N}^2, L^\infty([-T, T], \mathcal{X}_\varphi)} \leq C(T)(\|\partial_t \psi_N(t)u_0\|_{L_{\rho_N}^2, L^2([-T, T], \mathcal{X}_\varphi)} + \|\psi_N(t)u_0\|_{L_{\rho_N}^2, L^2([-T, T], \mathcal{X}_\varphi)}).$$

We can now exchange the norms in probability and in time by Fubini to get

$$\|u\|_{L_{\nu_N}^2, L^\infty([-T, T], \mathcal{X}_\varphi)} \leq C(T)(\|\partial_t \psi_N(t)u_0\|_{L^2([-T, T], (L_{\rho_N}^2, \mathcal{X}_\varphi))} + \|\psi_N(t)u_0\|_{L^2([-T, T], (L_{\rho_N}^2, \mathcal{X}_\varphi))}).$$

As ψ_N is the flow of (8), we get

$$\|\partial_t \psi_N(t)u_0\|_{L^2([-T, T], (L_{\rho_N}^2, \mathcal{X}_\varphi))} \leq \|\Delta \psi_N(t)u_0\|_{L^2([-T, T], (L_{\rho_N}^2, \mathcal{X}_\varphi))} + \|\chi_N |\psi_N(t)u_0|^2\|_{L^2([-T, T], (L_{\rho_N}^2, \mathcal{X}_\varphi))}.$$

We recall that \mathcal{Z}_φ is given by (4). Thanks to its L^2 part, we have

$$\|\Delta \psi_N(t)u_0\|_{L^2([-T, T], (L_{\rho_N}^2, \mathcal{X}_\varphi))} \leq \|\psi_N(t)u_0\|_{L^2([-T, T], (L_{\rho_N}^2, \mathcal{Z}_\varphi))}$$

and

$$\|\psi_N(t)u_0\|_{L^2([-T, T], (L_{\rho_N}^2, \mathcal{X}_\varphi))} \leq \|\psi_N(t)u_0\|_{L^2([-T, T], (L_{\rho_N}^2, \mathcal{Z}_\varphi))}.$$

And thanks to its L^6 part and the fact that $\chi_N \leq 1$, we have

$$\|\chi_N |\psi_N(t)u_0|^2\|_{L^2([-T, T], (L_{\rho_N}^2, \mathcal{X}_\varphi))} \leq \|\psi_N(t)u_0\|_{L^2([-T, T], (L_{\rho_N}^2, \mathcal{Z}_\varphi))}.$$

Therefore we get

$$\|u\|_{L_{\nu_N}^2, L^\infty([-T, T], \mathcal{X}_\varphi)} \leq C(T)\|\psi_N(t)u_0\|_{L^2([-T, T], (L_{\rho_N}^2, \mathcal{Z}_\varphi))}.$$

We now use the invariance of ρ_N under $\psi_N(t)$ for the topological σ -algebra of \mathcal{Z}_φ to get

$$\|u\|_{L_{\nu_N}^2, L^\infty([-T, T], \mathcal{X}_\varphi)} \leq C(T)\|u_0\|_{L^2([-T, T], (L_{\rho_N}^2, \mathcal{Z}_\varphi))}$$

and we take the norm in time to get

$$\|u\|_{L_{\nu_N}^2, L^\infty([-T, T], \mathcal{X}_\varphi)} \leq C(T)\sqrt{T}\|u_0\|_{L_{\rho_N}^2, \mathcal{Z}_\varphi}$$

which concludes the proof of the first lemma. \square

We are left with proving that there exists φ such that $\|u\|_{L_{\rho_N}^2, \mathcal{Z}_\varphi}$ is uniformly bounded in N .

We divide the problem into two parts by writing \mathcal{Z}_φ as $\mathcal{Z}_\varphi^2 \cap \mathcal{Z}_\varphi^6$ with \mathcal{Z}_φ^2 the space induced by the norm

$$\|\langle x \rangle^{-2}(1 + \varphi)^{-1}D^{\sigma+2}f\|_{L^2}$$

and \mathcal{Z}_φ^6 the space induced by the norm

$$\|\langle x \rangle^{-2}(1 + \varphi)^{-1/3}f\|_{L^6}.$$

We start with the L^6 part as the absence of derivatives makes it easier to deal with.

Lemma 4.10. *Let φ be a non negative, even function increasing on \mathbb{R}^+ such that $\varphi \geq \varphi_6^{1/2}$ where φ_6 is the one defined in (18), then*

$$\|u\|_{L_{\rho_N}^2, \mathcal{Z}_\varphi^6}$$

is uniformly bounded in N .

Proof. As ρ_N is a probability measure we have $\|\cdot\|_{L_{\rho_N}^2} \leq \|\cdot\|_{L_{\rho_N}^6}$, hence we have

$$\|u\|_{L_{\rho_N}^2, \mathcal{Z}_\varphi^6}^6 \leq \mathbb{E}_{\rho_N} \left(\int_{\mathbb{R}} dx \langle x \rangle^{-12} (1 + \varphi(x))^{-2} |u(x)|^6 \right).$$

We exchange the two integrations to get

$$\|u\|_{L_{\rho_N}^2, \mathcal{Z}_\varphi^6}^6 \leq \int_{\mathbb{R}} dx \langle x \rangle^{-12} (1 + \varphi(x))^{-2} \mathbb{E}_{\rho_N} (|u(x)|^6).$$

We use Lemma 4.5 to get

$$\mathbb{E}_{\rho_N} (|u(x)|^6) \leq \varphi_6(x)$$

which yields

$$\|u\|_{L_{\rho_N}^2, \mathcal{Z}_\varphi^6}^6 \leq \int_{\mathbb{R}} dx \langle x \rangle^{-12} (1 + \varphi(x))^{-2} \varphi_6(x).$$

With the choice of φ , this integral converge and does not depend on N . \square

We now deal with the L^2 part of \mathcal{Z}_φ .

Lemma 4.11. *Let $s < \frac{1}{4}$. Let ξ be a smooth positive even function decreasing on \mathbb{R}_+ and flat enough in $+\infty$ in the sense that*

- $|D^s \xi(x)|^2 \lesssim \varphi_2(x)^{-1} \langle x \rangle^{-2}$,
- $|\xi(x)|^2 \lesssim \varphi_{2,s}(x)^{-1} \langle x \rangle^{-3}$,
- $|\xi(x)|^{1-2s} \lesssim \varphi_2(x)^{-1} \langle x \rangle^{-2}$,

where φ_2 and $\varphi_{2,s}$ are the functions defined in Lemma 4.5.

Then we get that

$$\|\xi(x) D^s u\|_{L_{\rho_N}^2, L^2(\mathbb{R})}$$

is uniformly bounded in N .

Proof. We have

$$\|\xi(x) D^s u\|_{L_{\rho_N}^2, L^2(\mathbb{R})} \leq I + II$$

with

$$I = \|(D^s \xi)(x) u\|_{L_{\rho_N}^2, L^2(\mathbb{R})}, \quad II = \|D^s(\xi(x) u)\|_{L_{\rho_N}^2, L^2(\mathbb{R})}.$$

Let us start with I . We have

$$I^2 = \mathbb{E}_{\rho_N} \left(\int dx |D^s \xi(x)|^2 |u(x)|^2 \right)$$

and we exchange the integrals to get

$$I^2 = \int dx |D^s \xi(x)|^2 \mathbb{E}_{\rho_N} (|u(x)|^2).$$

We use the fact that by Lemma 4.5 we have $\mathbb{E}_{\rho_N}(|u(x)|^2) \leq \varphi_2(x)$ and our assumptions on ξ to make the integral converge and to get that I is uniformly bounded in N .

The quantity II can be written as

$$II^2 = \mathbb{E}_{\rho_N} \left(\int_{\mathbb{R} \times \mathbb{R}} dx dy \frac{|\xi(x)u(x) - \xi(y)u(y)|^2}{|x - y|^{1+2s}} \right).$$

We use symmetry over x and y to get

$$II^2 = 2\mathbb{E}_{\rho_N} \left(\int_{|x| \leq |y|} dx dy \frac{|\xi(x)u(x) - \xi(y)u(y)|^2}{|x - y|^{1+2s}} \right).$$

We now divide II^2 into two parts as $II^2 \leq A + B$ with

$$A = 2\mathbb{E}_{\rho_N} \left(\int_{|x| \leq |y|} dx dy \frac{|\xi(x) - \xi(y)|^2}{|x - y|^{1+2s}} |u(x)|^2 \right)$$

and

$$B = 2\mathbb{E}_{\rho_N} \left(\int_{|x| \leq |y|} dx dy \frac{|u(x) - u(y)|^2}{|x - y|^{1+2s}} |\xi(y)|^2 \right).$$

We exchange the order of integration to get

$$A = 2 \int_{|x| \leq |y|} dx dy \frac{|\xi(x) - \xi(y)|^2}{|x - y|^{1+2s}} \mathbb{E}_{\rho_N}(|u(x)|^2)$$

and

$$B = 2 \int_{|x| \leq |y|} dx dy \mathbb{E}_{\rho_N} \left(\frac{|u(x) - u(y)|^2}{|x - y|^{1+2s}} \right) |\xi(y)|^2.$$

We use (18)-(19) (notice that the couple $(r, \alpha) = (2, s)$ falls within the assumptions of Lemma 4.5) to get

$$A = 2 \int_{|x| \leq |y|} dx dy \frac{|\xi(x) - \xi(y)|^2}{|x - y|^{1+2s}} \varphi_2(x)$$

and as $|x| \leq |y|$,

$$B = 2 \int_{|x| \leq |y|} dx dy \varphi_{2,s}(y) |\xi(y)|^2.$$

For B , we integrate in x to get

$$B = 4 \int_{\mathbb{R}} dy |y| \varphi_{2,s}(y) |\xi(y)|^2$$

and we use the hypothesis on ξ to get this integral converge and is uniformly bounded in N .

For A , we use the smoothness and flatness of ξ at ∞ to get that ξ' is bounded and hence for $|x - y| \leq 1$

$$\frac{|\xi(x) - \xi(y)|^2}{|x - y|^{1+2s}} \lesssim (|\xi(x)| + |\xi(y)|)^{2-1-2s}$$

and the fact that ξ is even, decreasing on \mathbb{R}^+ and $|x| \leq |y|$ to get

$$\frac{|\xi(x) - \xi(y)|^2}{|x - y|^{1+2s}} \lesssim |\xi(x)|^{1-2s}.$$

When $|x - y| \geq 1$ we get

$$\frac{|\xi(x) - \xi(y)|^2}{|x - y|^{1+2s}} \leq \frac{|\xi(x)|^2}{|x - y|^{1+2s}}.$$

Therefore, we have

$$A \lesssim \int_{|x| \leq |y|, |x-y| \leq 1} dx dy |\xi(x)|^{1-2s} \varphi_2(x) + \int_{|x| \leq |y|, |x-y| \geq 1} dx dy \frac{|\xi(x)|^2}{|x - y|^{1+2s}} \varphi_2(x).$$

We drop the restriction $|x| \leq |y|$ and we integrate in y . We have that $\int_{|x-y| \leq 1} dy$ is finite and does not depend on x and so is $\int_{|x-y| \geq 1} \frac{dy}{|x-y|^{1+2s}}$, hence

$$A \lesssim \int_{\mathbb{R}} dx |\xi(x)|^{1-2s} \varphi_2(x) + \int_{\mathbb{R}} dx |\xi(x)|^2 \varphi_2(x)$$

and we use the assumptions on ξ to get that the integral converges and are uniformly bounded in N . \square

Proof of Proposition 4.8. By Lemma 4.9, we have that it is sufficient to get a φ such that

$$\|u\|_{L^2_{\rho_N}, \mathcal{Z}_\varphi} \leq C$$

where the constant C does not depend on N to conclude. We take $\varphi \leq \varphi_6^{1/2}$ and such that $(1 + \varphi(x))\langle x \rangle^2 = \xi(x)^{-1}$ with ξ satisfying the assumptions of Lemma 4.11. Then, by Lemma 4.10 and Lemma 4.11, we get that $\|u\|_{L^2_{\rho_N}, \mathcal{Z}_\varphi}$ is uniformly bounded in N which concludes the proof. \square

We are now ready to prove the main result of this section.

Proposition 4.12. *Let $T > 0$ and $\varepsilon > 0$. Then the family of measures $(\nu_N)_{N \geq 1}$ is tight in $\mathcal{X}_{T,\varphi}^\varepsilon$.*

Proof. Let $\delta > 0$, and define the set

$$K_\delta := \{u \in \mathcal{X}_{T,\varphi}^\varepsilon : \|u\|_{\mathcal{X}_{T,\varphi}} \leq \delta^{-1}\}.$$

Since the embedding $\mathcal{X}_{T,\varphi} \subset \mathcal{X}_{T,\varphi}^\varepsilon$ is compact (see Corollary 4.7), we have that the set K_δ is compact in $\mathcal{X}_{T,\varphi}^\varepsilon$ for any $\varepsilon > 0$. Moreover, thanks to (23) and Hölder inequality, we have

$$\nu_N(K_\delta^c) \leq \delta \|u\|_{L^1_{\nu_N} \mathcal{X}_{T,\varphi}} \leq \delta C.$$

Therefore, the family of measures $(\nu_N)_{N \geq 1}$ is tight in $\mathcal{X}_{T,\varphi}^\varepsilon$. \square

4.4 Existence of a weak flow for NLS

In this subsection, we use Skorokhod's theorem to prove the existence of a weak flow for NLS.

We apply Skorokhod's theorem to get the following proposition.

Proposition 4.13. *Up to a subsequence, there exists a probability space (Ω, \mathcal{F}, P) and a sequence of random variables $(X_N)_N$ with values in $\mathcal{X}_{\varphi,T}^\varepsilon$ such that the law of X_N is ν_N and X_N converges almost surely in $\mathcal{X}_{\varphi,T}^\varepsilon$ towards a random variable X . Besides, for all $t \in \mathbb{R}$, almost surely, we have $X_N(t) = \psi_N(t)Y_N$ with $Y_N = X_N(t=0)$ and the law of $X_N(t)$ is ρ_N .*

Proof. This is a direct application of Skorokhod's theorem, as explained in Subsection 3.3 and of Propositions 3.13 and 3.14. \square

Proposition 4.14. *The law of $X(t)$ is ρ .*

Proof. We have that X_N almost surely converges towards X in $\mathcal{X}_{\varphi,T}^\varepsilon = C([-T, T], \mathcal{X}_\varphi^\varepsilon)$. Hence for all $t \in [-T, T]$, $X_N(t)$ almost surely converges in $\mathcal{X}_\varphi^\varepsilon$ towards $X(t)$. The almost sure convergence implies the convergence in law. Hence, the law of $X(t)$ in $\mathcal{X}_\varphi^\varepsilon$ is the limit of ρ_N in $\mathcal{X}_\varphi^\varepsilon$, that is ρ . \square

Proposition 4.15. *The random variable X is almost surely a weak solution to*

$$i\partial_t u = -\Delta u + |u|^2 u.$$

Proof. We have that X_N is almost surely a solution in $\mathcal{X}_{\varphi,T}^\varepsilon$ of

$$i\partial_t X_N + \Delta X_N - \chi_N |X_N|^2 X_N = 0.$$

Since almost surely X_N converges in $\mathcal{X}_{\varphi,T}^\varepsilon$ towards X , $i\partial_t X_N$ converges in the sense of distribution towards $i\partial_t X$, and ΔX_N towards ΔX .

We explain why for almost all $\omega \in \Omega$, there exists a subsequence $\chi_{N_k} |X_{N_k}(\omega)|^2 X_{N_k}(\omega)$ which converges towards $|X(\omega)|^2 X(\omega)$ by proving that for some φ , $(1 + \varphi)^{-1} \langle x \rangle^{-2} X_N$ converges towards $(1 + \varphi)^{-1} \langle x \rangle^{-2} X$ in $L^r(\Omega \times [-T, T] \times \mathbb{R})$ for all $r \in]1, \infty[$.

We recall that χ_N converges towards 1 in the norm $\|\langle x \rangle^{-1} \cdot\|_{L^\infty}$ by construction.

With the same techniques as in Subsection 4.3, given that the law of $X_N(t)$ is ρ_N and the law of $X(t)$ is ρ , we have that for $s \leq \frac{2}{r}$, $s \leq 1 - \frac{3}{2r}$,

$$\|(1 + \varphi)^{-1} \langle x \rangle^{-1} D^s X_N\|_{L^r(\Omega \times [-T, T] \times \mathbb{R})}$$

is uniformly bounded in N and that

$$\|\langle x \rangle^{-1} (1 + \varphi)^{-1} D^s X\|_{L^r(\Omega \times [-T, T] \times \mathbb{R})}$$

is finite for all $s < \frac{1}{2}$ and $r \in]1, \infty[$. Let

$$C = \max_N (\sup \|\langle x \rangle^{-1} (1 + \varphi)^{-1} D^s X_N\|_{L^r(\Omega \times [-T, T] \times \mathbb{R})}, \|\langle x \rangle^{-1} (1 + \varphi)^{-1} D^s X\|_{L^r(\Omega \times [-T, T] \times \mathbb{R})}).$$

We have

$$\begin{aligned} \|\langle x \rangle^{-2} (1 + \varphi)^{-1} (X - X_N)\|_{L^r(\Omega \times [-T, T] \times \mathbb{R})} &\leq \\ &\langle R \rangle^{-1} 2C + M^{-s} \langle R \rangle (1 + \varphi(R)) 2C + \|\Pi_M 1_{|x| \leq R} (X - X_N)\|_{L^r(\Omega \times [-T, T] \times \mathbb{R})}. \end{aligned}$$

Since $\Pi_M 1_{|x| \leq R}$ projects into a space of finite dimension, we get that $\mathcal{X}_\varphi^\varepsilon$ and $L^r(\mathbb{R})$ have equivalent topologies on this space, which yields

$$\|\Pi_M 1_{|x| \leq R} (X - X_N)\|_{L^r(\mathbb{R})} \leq C(M, R) \|X - X_N\|_{\mathcal{X}_\varphi^\varepsilon}$$

and thus

$$\|\Pi_M 1_{|x| \leq R} (X - X_N)\|_{L^r(\Omega \times [-T, T] \times \mathbb{R})} \leq C(M, R) \|\Pi_M 1_{|x| \leq R} (X - X_N)\|_{L^r(\Omega \times [-T, T] \times \mathbb{R}), \mathcal{X}_\varphi^\varepsilon}$$

Since $\|X_N\|_{L^r(\Omega \times [-T, T], \mathcal{X}_\varphi^\varepsilon)}$ is uniformly bounded in N and $\|X - X_N\|_{L^r([-T, T], \mathcal{X}_\varphi^\varepsilon)} \leq T^{1/r} \|X - X_N\|_{\mathcal{X}_{\varphi, T}^\varepsilon}$ converges towards 0, we get by the dominated convergence theorem that

$$\|\Pi_M 1_{|x| \leq R}(X - X_N)\|_{L^r(\Omega \times [-T, T] \times \mathbb{R})} \rightarrow 0$$

and therefore, so does $\|\langle x \rangle^{-2}(1 + \varphi)^{-1}(X - X_N)\|_{L^r(\Omega \times [-T, T] \times \mathbb{R})}$. With $r = 6$, we get that $|X_N|^2 X_N$ converges towards $|X|^2 X$ in $\langle x \rangle^6 (1 + \varphi)^3 L^2(\Omega \times [-T, T] \times \mathbb{R})$ and hence that $\chi_N |X_N|^2 X_N$ converges towards $|X|^2 X$ in $\langle x \rangle^7 (1 + \varphi)^3 L^2(\Omega \times [-T, T] \times \mathbb{R})$. We deduce from that that for almost all $\omega \in \Omega$, there exists a subsequence $X_{N_k}(\omega)$ such that $\chi_{N_k} |X_{N_k}|^2 X_{N_k}$ converges towards $|X|^2 X$ in $\langle x \rangle^7 L^2([-T, T] \times \mathbb{R})$ and hence weakly.

Thus, for almost all $\omega \in \Omega$, there exists a subsequence $X_{N_k}(\omega)$ such that $i\partial_t X_{N_k}(\omega) + \Delta X_{N_k}(\omega) - \chi_{N_k} |X_{N_k}(\omega)|^2 X_{N_k}(\omega)$ goes to $i\partial_t X(\omega) + \Delta X(\omega) - |X(\omega)|^2 X(\omega)$, which ensures that almost surely and in the sense of distributions

$$i\partial_t X + \Delta X - |X|^2 X = 0.$$

□

This concludes the proof of the main theorem.

Definition 4.16. Let Ω' be the set of $\omega \in \Omega$ such that $X(\omega)$ satisfies $i\partial_t u = -\Delta u + |u|^2 u$. Let A be the image by $X(t = 0)$ of Ω' . For all $u_0 \in A$, let

$$\psi(t)(u_0) = \{X(t)(\omega) \mid \omega \in \Omega' \cap X(0)^{-1}(\{u_0\})\}.$$

This defines a weak flow $\psi(t)$ of $i\partial_t u = -\Delta u + |u|^2 u$. In particular, we do not have uniqueness of the solution.

4.5 Invariance of ρ under the weak flow, further remarks

In this subsection, we interpret $\psi(t)$ and $X(t)$ in terms of measures.

Definition 4.17. Let $t \in \mathbb{R}$, we call \mathcal{F}_t the set of measurable sets A of $\mathcal{X}_\varphi^\varepsilon$ such that for all $u_0 \in \mathcal{X}_\varphi^\varepsilon$, if $\psi(t)(u_0) \cap A \neq \emptyset$ then $\psi(t)(u_0) \subseteq A$.

Proposition 4.18. The set \mathcal{F}_t is a σ -algebra included in the topological σ -algebra of $\mathcal{X}_\varphi^\varepsilon$.

Proof. The empty set belongs to \mathcal{F}_t .

Let $A \in \mathcal{F}_t$ and A^c its complementary. Let $u_0 \in \mathcal{X}_\varphi^\varepsilon$.

If $\psi(t)(u_0)$ is not included in A^c then, we have that $\psi(t)(u_0) \cap A$ is not empty. Hence, as A belongs to \mathcal{F}_t , we get that $\psi(t)(u_0)$ is included in A and thus $\psi(t)u_0 \cap A^c = \emptyset$.

The converse statement is that if $\psi(t)u_0 \cap A^c$ is not empty then $\psi(t)u_0$ is included in A^c and hence A^c belongs to \mathcal{F}_t .

Let $(A_n)_{n \in \mathbb{N}}$ be a sequence of sets of \mathcal{F}_t and let $A = \bigcup A_n$. Let $u_0 \in \mathcal{X}_\varphi^\varepsilon$.

If $\psi(t)(u_0) \cap A$ is different from the empty set then there exists $n \in \mathbb{N}$ such that $\psi(t)(u_0) \cap A_n$ is non empty. Hence, $\psi(t)u_0 \subseteq A_n \subseteq A$. Thus $A \in \mathcal{F}_t$. □

Remark 4.2. The σ -algebra \mathcal{F}_t may be trivial. Indeed, if $\psi(t)(u_0)$ is either equal to the empty set or the full set then \mathcal{F}_t is trivial.

Nevertheless, let $A_0 = \{u_0 \in \mathcal{X}_\varphi^\varepsilon \mid \text{Card}(\psi(t)(u_0)) = 1\}$ and assume that there exists A_t measurable such that A_t is included in $\psi(t)(A_0)$ then \mathcal{F}_t contains at least all the $A_t \cap A$ with A measurable in $\mathcal{X}_\varphi^\varepsilon$.

Remark 4.3. Let us comment upon the lack of uniqueness of the flow. Assume that the cardinal of $\psi(t)(u_0)$ is strictly more than 1. Then, there exists ω_1 and ω_2 in Ω such that $X(t)(\omega_1) \neq X(t)(\omega_2)$ but $X(0)(\omega_1) = X(0)(\omega_2) = u_0$. We recall that for all $\tau \in \mathbb{R}$ and $i = 1, 2$, $X(\tau)(\omega_i)$ is the limit of $X_N(\tau)(\omega_i)$. Since $X(t)(\omega_1) \neq X(t)(\omega_2)$ we get that there do not exist subsequences such that

$$\psi_{N_k}(t)(X_{N_k}(0)(\omega_1)) = X_{N_k}(t)(\omega_1) = X_{N'_k}(t)(\omega_2) = \psi_{N'_k}(t)(X_{N'_k}(0)(\omega_2))$$

and because of the uniqueness and reversibility of ψ_{N_k} that there do not exist subsequences such that

$$X_{N_k}(0)(\omega_1) = X_{N'_k}(0)(\omega_2).$$

In other words, $X_N(0)(\omega_1)$ has to converge in a different way to u_0 from $X_N(0)(\omega_2)$.

Hence, if one could prove that almost surely $X(0)(\omega_1) = X(0)(\omega_2)$ implies for example that $X_N(0)(\omega_1) = X_N(0)(\omega_2)$ for an infinite number of N s then one would get uniqueness of the flow. To us, it is not obvious how to prove this or even if this is true, but we expect that if it is possible, one should understand it at the level of the convergence of ρ_N towards ρ .

Definition 4.19. Let $A \subseteq X_\varphi^\varepsilon$ and $t \in \mathbb{R}$, we call the reverse image of A by $\psi(t)$ the set

$$\psi(t)^{-1}(A) = \{u_0 \in X_\varphi^\varepsilon \mid \psi(t)(u_0) \in A\}.$$

Proposition 4.20. Let $t \in \mathbb{R}$ and $A \in \mathcal{F}_t$, we have

$$X(0)^{-1}(\psi(t)^{-1}(A)) = X(t)^{-1}(A).$$

Proof. Let $\omega \in \Omega$. We have that ω belongs to $X(0)^{-1}(\psi(t)^{-1}(A))$ if and only if $\psi(t)(X(0)(\omega)) \subset A$. But since A belongs to \mathcal{F}_t then $\psi(t)(X(0)(\omega)) \subset A$ is equivalent to $X(t)(\omega) \in A$. Indeed, $X(t)(\omega)$ belongs to $\psi(t)(X(0)(\omega))$. Therefore, $\omega \in X(0)^{-1}(\psi(t)^{-1}(A))$ is equivalent to $\omega \in X(t)^{-1}(A)$ which concludes the proof. \square

Definition 4.21. Define ρ^t the transported measure of ρ under $\psi(t)$ on \mathcal{F}_t as

$$\rho^t(A) = \rho(\psi(t)^{-1}(A)) := P(X(0)^{-1}(\psi(t)^{-1}(A))).$$

Proposition 4.22. For all $A \in \mathcal{F}_t$,

$$\rho^t(A) = \rho(A).$$

Proof. With the last proposition

$$\rho^t(A) = P(X(t)^{-1}(A))$$

and the law of $X(t)$ is ρ . \square

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